

# Classification and stability of simple homoclinic cycles in $\mathbb{R}^5$

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## Abstract

We study asymptotic stability of simple homoclinic cycles in equivariant dynamical systems. We introduce a classification of simple homoclinic cycles in  $\mathbb{R}^n$  based on the action of the system symmetry group. For systems in  $\mathbb{R}^5$ , we list all classes of simple homoclinic cycles. For each class, we derive necessary and sufficient conditions for asymptotic stability and fragmentary asymptotic stability in terms of eigenvalues of linearisation near the steady state involved in the cycle.

## 1 Introduction

In this paper we continue the investigation [20] of asymptotic stability of structurally stable heteroclinic cycles in a smooth dynamical system

$$\dot{\mathbf{x}} = f(\mathbf{x}), \quad f : \mathbb{R}^n \rightarrow \mathbb{R}^n \quad (1)$$

equivariant under the action of a non-trivial finite symmetry group  $\Gamma$ :

$$f(\gamma\mathbf{x}) = \gamma f(\mathbf{x}), \quad \text{for all } \gamma \in \Gamma \subset \mathbf{O}(n). \quad (2)$$

Let  $\xi_1, \dots, \xi_m \in \mathbb{R}^n$  be hyperbolic equilibria of (1) and  $\kappa_j : \xi_j \rightarrow \xi_{j+1}$ ,  $j = 1, \dots, m$ ,  $\xi_{m+1} = \xi_1$ , be a set of trajectories from  $\xi_j$  to  $\xi_{j+1}$ . The union of the equilibria and the connecting trajectories is called a *heteroclinic cycle*. A heteroclinic cycle is *structurally stable* (or robust), if for each  $j$  there exists an invariant subspace  $P_j$  of (1) such that  $\kappa_j$  belongs to this subspace, and in this subspace  $\xi_{j+1}$  is a sink [2, 13, 25]. Cycles where all connections are one-dimensional are called *simple heteroclinic cycles*.

Heteroclinic cycles often arise in systems related to biology [17, 11], fluid dynamics [4, 10, 24] and game theory [3]. Heteroclinic cycles are often responsible for complex and intermittent behaviour [13]. They may have simple geometric structure but complex local attraction properties — heteroclinic cycles which are not asymptotically stable can attract a positive measure set in its small neighbourhood [3, 5, 7, 12, 16, 22]. In [20] such cycles were called fragmentarily asymptotically stable.

The theory of asymptotic and fragmentary asymptotic stability is yet incomplete even for simple heteroclinic cycles. A sufficient condition for asymptotic stability of heteroclinic cycles was presented in [14]. Necessary and sufficient conditions for asymptotic stability of simple homoclinic and heteroclinic cycles in  $\mathbb{R}^4$  were proven in [6, 8, 16], for fragmentary asymptotic stability of simple heteroclinic cycles in  $\mathbb{R}^4$  in [21]. Asymptotic stability of heteroclinic cycles in some particular systems was studied in [3, 5, 6, 7, 8, 12, 15, 22, 23]. Necessary and sufficient conditions for asymptotic stability of the so-called type A cycles were proven in [14], and for asymptotic and fragmentary asymptotic stability of the so-called type Z cycles in [20].

Here, we consider stability of simple homoclinic cycles. We introduce a classification of cycles that depends on the action of the system symmetry group. Some our classes are comprised of type A homoclinic cycles, some classes of type Z homoclinic cycles, the remaining classes of cycles of other types not studied so far. For cycles in  $\mathbb{R}^5$  we formulate necessary and sufficient conditions for stability in terms of eigenvalues of the linearisation near the steady state.

## 2 Definitions

### 2.1 Stability

Let us recall the definitions of various types of asymptotic stability of an invariant set of system (1). We denote by  $\Phi_t(\mathbf{x})$  the trajectory of (1) that starts at point  $\mathbf{x}$ . For a set  $X$  and a number  $\epsilon > 0$ , the  $\epsilon$ -neighbourhood of  $X$  is the set

$$B_\epsilon(X) = \{\mathbf{x} \in \mathbb{R}^n : d(\mathbf{x}, X) < \epsilon\}. \quad (3)$$

Let  $X$  be a compact invariant set of the system (1). We denote by  $\mathcal{B}_\delta(X)$  its  $\delta$ -local basin of attraction:

$$\mathcal{B}_\delta(X) = \{\mathbf{x} \in \mathbb{R}^n : d(\Phi_t(\mathbf{x}), X) < \delta \text{ for any } t \geq 0 \text{ and } \lim_{t \rightarrow \infty} d(\Phi_t(\mathbf{x}), X) = 0\}. \quad (4)$$

**Definition 1** A compact invariant set  $X$  is asymptotically stable, if for any  $\delta > 0$  there exists  $\epsilon > 0$  such that

$$B_\epsilon(X) \subset \mathcal{B}_\delta(X).$$

**Definition 2** A compact invariant set  $X$  is fragmentarily asymptotically stable, if for any  $\delta > 0$

$$\mu(\mathcal{B}_\delta(X)) > 0.$$

(Here  $\mu$  denotes the Lebesgue measure in  $\mathbb{R}^n$ .)

**Definition 3** A compact invariant set  $X$  is completely unstable, if there exists  $\delta > 0$  such that  $\mu(\mathcal{B}_\delta(X)) = 0$ .

We define now various types of stability of a fixed point  $\mathbf{x}$  of a map  $g : \mathbb{R}^n \rightarrow \mathbb{R}^n$  (i.e.  $g\mathbf{x} = \mathbf{x}$ ).

**Definition 4** A fixed point  $\mathbf{x} \in \mathbb{R}^n$  of a map  $g : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is asymptotically stable, if for any  $\delta > 0$  there exists  $\epsilon > 0$ , such that  $|\mathbf{y} - \mathbf{x}| < \epsilon$  implies

$$|g^k \mathbf{y} - \mathbf{x}| < \delta \text{ for all } k > 0 \text{ and } \lim_{k \rightarrow \infty} g^k \mathbf{y} = \mathbf{x}.$$

**Definition 5** ([20], adapted). A fixed point  $\mathbf{x} \in \mathbb{R}^n$  of a map  $g : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is fragmentarily asymptotically stable, if for any  $\delta > 0$  the measure of the set

$$V_\delta(\mathbf{x}) =: \{\mathbf{y} \in \mathbb{R}^n : |g^k \mathbf{y} - \mathbf{x}| < \delta \text{ for all } k > 0 \text{ and } \lim_{k \rightarrow \infty} g^k \mathbf{y} = \mathbf{x}\}$$

is positive.

**Definition 6** A fixed point  $\mathbf{x} \in \mathbb{R}^n$  of a map  $g : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is completely unstable, if there exists  $\delta > 0$  such that

$$\mu(V_\delta(\mathbf{x})) = 0.$$

## 2.2 Heteroclinic cycles

Let  $\xi_1, \dots, \xi_m$  be hyperbolic equilibria of the system (1), (2) with stable and unstable manifolds  $W^s(\xi_j)$  and  $W^u(\xi_j)$ , respectively. Assuming  $\xi_{m+1} = \xi_1$ , we denote by  $\kappa_j$ ,  $j = 1, \dots, m$ , the set of trajectories from  $\xi_j$  to  $\xi_{j+1}$ :  $\kappa_j = W^u(\xi_j) \cap W^s(\xi_{j+1}) \neq \emptyset$ .

**Definition 7** A heteroclinic cycle is a connected component of the group orbit of the equilibria  $\{\xi_1, \dots, \xi_m\}$  and their connecting orbits  $\{\kappa_1, \dots, \kappa_m\}$  under the action of  $\Gamma$ .

The *isotropy group* of a point  $x \in \mathbb{R}^n$  is the subgroup of  $\Gamma$  satisfying

$$\Sigma_x = \{\gamma \in \Gamma : \gamma x = x\}.$$

The *fixed-point subspace* of a subgroup  $\Sigma \subset \Gamma$  is the subspace

$$\text{Fix}(\Sigma) = \{\mathbf{x} \in \mathbb{R}^n : \sigma \mathbf{x} = \mathbf{x} \text{ for all } \sigma \in \Sigma\}.$$

**Definition 8** A heteroclinic cycle is structurally stable (or robust), if for any  $j$ ,  $1 \leq j \leq m$ , there exist  $\Sigma_j \subset \Gamma$  and a fixed-point subspace  $P_j = \text{Fix}(\Sigma_j)$  such that

- $\xi_{j+1}$  is a sink in  $P_j$ ;
- $\kappa_j \subset P_j$ .

We denote  $L_j = P_{j-1} \cap P_j$ , by  $T_j$  the isotropy subgroup of  $L_j$  (evidently,  $\xi_j \in L_j$ ), and by  $P_j^\perp$  the orthogonal complement to  $P_j$  in  $\mathbb{R}^n$ .

For a structurally stable heteroclinic cycle, eigenvalues of  $df(\xi_j)$  can be divided into four sets [14, 15, 16]:

- Eigenvalues with the associated eigenvectors in  $L_j$  are called *radial*.
- Eigenvalues with the associated eigenvectors in  $P_{j-1} \ominus L_j$  are called *contracting*.
- Eigenvalues with the associated eigenvectors in  $P_j \ominus L_j$  are called *expanding*.
- Eigenvalues that do not belong to any of the three aforementioned groups are called *transverse*.

**Definition 9** ([16], adapted). A robust heteroclinic cycle in  $\mathbb{R}^n \setminus \{0\}$  is simple, if for any  $j$   $\dim(P_{j-1} \ominus L_j) = 1$ .

**Definition 10** [20] A simple robust heteroclinic cycle is of type Z, if for any  $j$

- $\dim P_j = \dim P_{j+1}$ ;
- the isotropy subgroup of  $P_j$ ,  $\Sigma_j$ , decomposes  $P_j^\perp$  into one-dimensional isotypic components.

Note that the first condition in the definition [20] of type Z cycles is redundant, because it is implied by the simplicity of the cycles:

**Lemma 1** Let  $X$  be a simple robust heteroclinic cycle, i.e. the respective fixed-point subspaces satisfy  $\dim(P_{j-1} \ominus L_j) = 1$  for any  $j$ ,  $1 \leq j \leq m$ . Then  $\dim P_j = \dim P_{j+1}$  for all  $j$ ,  $1 \leq j \leq m-1$ .

**Proof:** The dimension of the expanding eigenspace for a steady state  $\xi_j$  involved in a heteroclinic cycle can not be less than one, and therefore

$$\dim P_j \geq \dim L_j + 1 = \dim P_{j-1}.$$

This inequality applied for each  $m$  steady states yields

$$\dim P_1 \leq \dim P_2 \leq \dots \leq \dim P_m \leq \dim P_1$$

(recall that the equilibria are cyclically connected, i.e.,  $\xi_1 = \xi_{m+1}$ ). Here, the leftmost and rightmost values coincide, and thus all terms in the inequality are equal. **QED**

Since  $\dim(P_{j-1} \ominus L_j) = 1$  for a simple heteroclinic cycle, all expanding and contracting eigenspaces are one-dimensional.

**Definition 11** A simple robust heteroclinic cycle is of type A', if for any  $j$  the isotypic decomposition of  $P_j^\perp$  under the action of  $\Sigma_j$  involves only one isotypic component.

Our A' type cycles are a subset of type A cycles:

**Definition 12** [14, 15, 16] *A simple robust heteroclinic cycle is of type A, if for any  $j$*

- *all eigenvectors, associated with  $\lambda_j^c$ ,  $\lambda_j^t$ ,  $\lambda_{j+1}^e$  and  $\lambda_{j+1}^t$ , belong to the same isotypic component in the decomposition of  $P_j^\perp$  under  $\Sigma_j$ ;*
- *all eigenvectors of  $df(\xi_j)$ , associated with transverse eigenvalues with positive real parts, belong to the same isotypic component in the decomposition of  $P_j^\perp$  under  $\Sigma_j$ .*

Here  $\lambda_j^c$  and  $\lambda_j^t$  denote the contracting and transverse eigenvalues of  $df(\xi_j)$  with the minimum real parts, respectively, and  $\lambda_j^e$  the expanding eigenvalue with the maximum real part.

Note that if a system depends on a parameter and the classification involves conditions for eigenvalues, then on variation of the parameter the type of the cycle can change without any qualitative change in the overall behaviour of the system.<sup>1</sup> This is not the case, when the classification is based on the action of the symmetry group, as we propose here.

**Definition 13** *A heteroclinic cycle is called a homoclinic cycle, if there exists a symmetry  $\gamma \in \Gamma$  such that for any  $1 \leq j \leq m$*

$$\gamma \xi_j = \xi_{j+1}.$$

The symmetry  $\gamma$  is called a *twist* [2, 25].

In this paper we study stability of simple homoclinic cycles, and we use the symbols  $\xi$ ,  $\kappa$ ,  $P$  and  $L$  without indices. The radial eigenvalues of  $df(\xi)$  are denoted by  $-\mathbf{r} = \{-r_l\}$ ,  $1 \leq l \leq n_r$ , the contracting one by  $-c$ , the expanding one by  $e$  and the transverse ones by  $\mathbf{t} = \{t_l\}$ ,  $1 \leq l \leq n_t$ . Here  $n_r$  and  $n_t$  are the numbers of the radial and transverse eigenvalues, respectively.

### 3 Classification of simple homoclinic cycles in $\mathbb{R}^n$

By definition 8, a homoclinic cycle is structurally stable, if there exists a subgroup  $\Sigma \subset \Gamma$  such that the connection from  $\xi$  to  $\gamma\xi$  belongs to a fixed-point subspace  $P = \text{Fix}(\Sigma)$ . We introduce a classification of simple homoclinic cycles in  $\mathbb{R}^n$  that is based on the actions of  $\Sigma$  and  $\gamma$  (note that  $\gamma \notin \Sigma$ ) on  $P^\perp$ . It is coarser than those proposed in [25] for homoclinic cycles in  $\mathbb{R}^4$  and employed in [26, 7] for cycles in  $\mathbb{R}^4$  and  $\mathbb{R}^5$ . Nevertheless, our classification suffices to determine conditions for stability of homoclinic cycles in  $\mathbb{R}^5$  — for each class, they have the form of inequalities for eigenvalues of the linearisation  $df(\xi)$  (see table 2).

Let  $\mathbf{e}_k$ ,  $1 \leq k \leq K$ , be a basis in  $P^\perp$  comprised of eigenvectors of  $df(\xi)$ , and let  $\mathbf{h}_k$ ,  $1 \leq k \leq K$ , be a basis in  $P^\perp$  comprised of eigenvectors of  $df(\gamma\xi)$ . We assume that  $\mathbf{e}_1$  is the contracting eigenvector of  $df(\xi)$ ,  $\mathbf{h}_1$  is the expanding eigenvector of  $df(\gamma\xi)$ , and the remaining transverse eigenvectors are related:  $\mathbf{h}_k = \gamma\mathbf{e}_k$ ,  $2 \leq k \leq K$ . Suppose

$$P^\perp = U_1 \oplus \dots \oplus U_J \tag{5}$$

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<sup>1</sup>For instance, cycles of class 3-2-(12)(3) (see subsection 5.2 and table 2) are not of type A', but for  $t_1 < t_2$  they are of type A as defined in [14]. Such a cycle is asymptotically stable for  $c < e$  and  $t_1, t_2 < 0$ . When  $t_1$  exceeds  $t_2$ , the cycle just ceases to be of type A, although no bifurcations take place and the stability of the cycle does not change.

is the isotypic decomposition of  $P^\perp$  under the action of  $\Sigma$ . Let the eigenvectors in the basis be ordered in such a way that the  $\mathbf{h}_k$ , belonging to one isotypic component,  $U_j$ , (whose dimension we denote by  $l_j$ ), have consecutive indices,  $k = s + 1, \dots, s + l_j$ ; namely,  $U_1$  is spanned by  $\mathbf{h}_1, \dots, \mathbf{h}_{l_1}$ ,  $U_2$  by  $\mathbf{h}_{l_1+1}, \dots, \mathbf{h}_{l_1+l_2}$ , etc. Each isotypic component  $U_j$  is also spanned by the eigenvectors  $\mathbf{e}_k$  for some  $k = i_{L_{j-1}+1}, i_{L_{j-1}+2}, \dots, i_{L_j}$ , where we have denoted  $L_j = l_1 + \dots + l_j$ . We label such a heteroclinic cycle by a sequence of numbers, where subsequences associated with individual isotypic components are enclosed in parentheses:

$$(i_1, i_2, \dots, i_{l_1})(i_{l_1+1}, i_{l_1+2}, \dots, i_{L_2}) \dots (i_{L_{J-1}+1}, i_{L_{J-1}+2}, \dots, i_{L_J}).$$

All classes of homoclinic cycles in  $\mathbb{R}^5$  are listed in table 1. The sequences defined above are supplemented by two numbers: the dimension of  $P^\perp$  and the number of isotypic components (e.g. 3-1-(123) labels a cycle with a three-dimensional  $P^\perp$  comprised of a single isotypic component). Note that transverse eigendirections are interchangeable and the resultant cycles (e.g., 3-2-(2)(13) and 3-2-(3)(12)) are identical; thus, only one of them is listed.

## 4 Poincaré maps for homoclinic cycles in $\mathbb{R}^5$

Following [12, 16, 21], in order to examine stability of a homoclinic cycle we consider a Poincaré map near the cycle. In subsection 2.2 we have defined radial, contracting, expanding and transverse eigenvalues of the linearisation  $df(\xi)$ . Let  $(\tilde{\mathbf{u}}, \tilde{v}, \tilde{w}, \tilde{\mathbf{z}})$  be coordinates in the coordinate system with the origin at  $\xi$  and the basis comprised of the associated eigenvectors in the following order: radial, contracting, expanding and transverse.

If  $\tilde{\delta}$  is small, in a  $\tilde{\delta}$ -neighbourhood of  $\xi$  the system (1) can be approximated by the linear system<sup>2</sup>

$$\begin{aligned} \dot{u}_l &= -r_l u_l, \quad 1 \leq l \leq n_r \\ \dot{v} &= -cv \\ \dot{w} &= ew \\ \dot{z}_l &= t_l z_l, \quad 1 \leq l \leq n_t. \end{aligned} \tag{6}$$

Here,  $(\mathbf{u}, v, w, \mathbf{z})$  denote the scaled coordinates  $(\mathbf{u}, v, w, \mathbf{z}) = (\tilde{\mathbf{u}}, \tilde{v}, \tilde{w}, \tilde{\mathbf{z}})/\tilde{\delta}$ .

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<sup>2</sup>We assume here that all eigenvalues are real. The system under consideration can have a pair of complex conjugate radial eigenvalues, if the dimension of the radial eigenspace is larger than one, or it can have a pair of complex conjugate transverse eigenvalues, if the homoclinic cycle is of the classes 3-1-(123) or 3-2-(1)(23). The radial eigenvalues are not relevant in the study of stability. If transverse eigenvalues are complex, the estimates

$$k_1(|z_1| + |z_2|)|w|^{-t/e} \leq |z_1| \leq K_1(|z_1| + |z_2|)|w|^{-t/e}, \quad k_2(|z_1| + |z_2|)|w|^{-t/e} \leq |z_2| \leq K_2(|z_1| + |z_2|)|w|^{-t/e},$$

where  $t = \text{Re}(t_1) = \text{Re}(t_2)$ , can be employed in the proofs of stability and instability, respectively, instead of the exact expressions

$$z_1 = z_1|w|^{-t_1/e}, \quad z_2 = z_2|w|^{-t_2/e}.$$

Since only the values of exponents are important in the proofs, in the conditions for stability (see table 2)  $t_i$  is replaced by  $\text{Re}(t_i)$ , and no other modifications are required.

Let  $(\mathbf{u}_0, v_0)$  be the point in  $\gamma^{-1}P$  where the trajectory  $\gamma^{-1}\kappa$  intersects with the sphere  $|\mathbf{u}|^2 + v^2 = 1$ , and  $\mathbf{q}$  be local coordinates in the hyperplane tangent to the sphere at the point  $(\mathbf{u}_0, v_0)$ . We consider two crossections:

$$\tilde{H}^{(out)} = \{(\mathbf{u}, v, w, \mathbf{z}) : |\mathbf{u}|, |v|, |\mathbf{z}| \leq 1, w = 1\}$$

and

$$\tilde{H}^{(in)} = \{(\mathbf{q}, w, \mathbf{z}) : |\mathbf{q}|, |w|, |\mathbf{z}| \leq 1\}.$$

Near  $\xi$ , trajectories of system (1) can be approximated by a local map (called the *first return map*)  $\tilde{\phi} : \tilde{H}^{(in)} \rightarrow \tilde{H}^{(out)}$  that associates a point, where a trajectory crosses  $\tilde{H}^{(out)}$  with the point, where the trajectory crossed  $\tilde{H}^{(in)}$ . The global map  $\tilde{\psi} : \tilde{H}^{(out)} \rightarrow \gamma\tilde{H}^{(in)}$  associates a point where a trajectory crosses  $\gamma\tilde{H}^{(in)}$  with the point where its previously crossed  $\tilde{H}^{(out)}$ . The Poincaré map is the superposition  $\tilde{g} = \tilde{\psi}\tilde{\phi}$ . The  $w$ - and  $\mathbf{z}$ -components of the map  $\tilde{g}$  are independent of  $\mathbf{q}$ : this was shown in [21, 20] for slightly different systems, but the proof can be trivially modified to serve the case considered here. Thus, one can define the map  $g(w, \mathbf{z})$  that is the restriction of the map  $\tilde{g}$  into the  $(w, \mathbf{z})$  subspace. The stability properties of fixed points of the maps  $\tilde{g}$  and  $g$  are identical; hence, stability of a cycle is determined by stability of the fixed point  $(w, \mathbf{z}) = \mathbf{0}$  of the map  $g$ .

Denote by  $\phi$  and  $\psi$  the restrictions of  $\tilde{\psi}$  and  $\tilde{\phi}$  into  $P^\perp$ . In the leading order, the map  $\phi$  is

$$\phi(w, \{z_s\}) = (v_0 w^{c/e}, \{z_s |w|^{-t_s/e}\}) \quad (7)$$

(we use the coordinates  $(w, \mathbf{z})$  in  $H^{(in)}$  and  $(v, \mathbf{z})$  in  $H^{(out)}$ ). Note that the local map is expressed by (7) for any homoclinic cycle of whichever class.

Expressions for global maps are different for different classes of homoclinic cycles. The conditions of lemma 2 (see appendix A) are satisfied for all simple homoclinic cycles in  $\mathbb{R}^5$  except for the 3-2-(1)(23) cycles. We do not consider the 3-2-(1)(23) cycles henceforth in this section; the global map for them is derived in appendix A. By the lemma, each isotypic component of  $P^\perp$  is an invariant subspace of the map  $\psi$ .

Generically, in the leading order the global map  $\psi$  is linear in each isotypic component. The matrix,  $C$ , of the linear map

$$(w^{\gamma\xi}, \mathbf{z}^{\gamma\xi}) = \psi(v^\xi, \mathbf{z}^\xi) = C \begin{pmatrix} v^\xi \\ \mathbf{z}^\xi \end{pmatrix}, \quad (8)$$

(here superscripts indicate whether the components are in the basis of eigenvectors of  $df(\xi)$  or of  $df(\gamma\xi)$ ) is the product  $C = BA$ , where  $A$  is the matrix of the map  $\psi$  in the basis of eigenvectors of  $df(\xi)$ , and  $B$  is the matrix of transformation of  $(v^\xi, \mathbf{z}^\xi)$  into  $(w^{\gamma\xi}, \mathbf{z}^{\gamma\xi})$ . In the study of stability, we focus on the location of blocks in  $C = \{c_{ij}\}$  that vanish because the map  $\psi$  has invariant subspaces. Generically,  $c_{ij} \neq 0$ , if  $\mathbf{e}_i$  and  $\mathbf{h}_j$  belong to the same isotypic component in the decomposition (5). Hence, the location of zero entries of  $C$  can be determined applying the classification presented in section 3. Below we list exhaustively the possible forms of matrices  $C$  (where non-zero entries are shown by  $*$ ) for various types of homoclinic cycles in  $\mathbb{R}^5$ , and determine the general forms of Poincaré maps (see table 1).

If the subspace  $P^\perp$  is one-dimensional, then  $C$  is an  $1 \times 1$  matrix and the Poincaré map is just  $g(w) = c_{11}w^{c/e}$ .

If the subspace  $P^\perp$  is two-dimensional, then the classification of simple homoclinic cycles in  $\mathbb{R}^4$  is applicable, since for such cycles  $P^\perp$  is two-dimensional. Alternatively, note that such cycles are either of type  $A'$  (if the decomposition of  $P^\perp$  under  $\Sigma$  has only one isotypic component) or  $Z$  (if there are two components). In the former case the cycle is classified as 2-1-(12), generically none entries of its matrix  $C$  vanish, and thus the Poincaré map is  $g(w, z) = (c_{11}w^{c/e} + c_{12}z|w|^{-t/e}, c_{21}w^{c/e} + c_{22}z|w|^{-t/e})$ . In the latter case the cycles are of the 2-2-(1)(2) or 2-2-(2)(1) classes, the matrices of the global map  $\psi$  are

$$\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \quad \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix} \quad (9)$$

2-2-(1)(2)      2-2-(2)(1)

and the Poincaré maps are  $g(w, z) = (c_{11}w^{c/e}, c_{22}z|w|^{-t/e})$  (for the 2-2-(1)(2) cycle) and  $g(w, z) = (c_{12}z|w|^{-t/e}, c_{21}w^{c/e})$  (for the 2-2-(2)(1) cycle).

If the subspace  $P^\perp$  is three-dimensional, then homoclinic cycles are of types  $A'$  (if the decomposition (5) involves just one isotypic component),  $Z$  (if the decomposition involves three components), or of other types not studied so far. For a type  $A'$  cycle listed in tables 1 and 2 as 3-1-(123), generically all entries of  $C$  are non-zero. For type  $Z$  cycles,  $A$  is a diagonal matrix and  $B$  is a permutation matrix (provided vectors  $\mathbf{e}_j$  and  $\mathbf{h}_j$  in the bases are normalised), and hence  $C$  has one of the following forms:

$$\begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix} \quad \begin{pmatrix} * & 0 & 0 \\ 0 & 0 & * \\ 0 & * & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & * & 0 \\ * & 0 & 0 \\ 0 & 0 & * \end{pmatrix} \quad \begin{pmatrix} 0 & * & 0 \\ 0 & 0 & * \\ * & 0 & 0 \end{pmatrix} \quad (10)$$

3-3-(1)(2)(3)      3-3-(1)(3)(2)      3-3-(2)(1)(3)      3-3-(2)(3)(1)

Now suppose decomposition (5) involves two isotypic components. Then the cycles are either 3-2-( $\cdot$ )( $\cdot$ ), or 3-2-( $\cdot$ )( $\cdot$ ), where dots should be replaced by a permutation of indices  $\{1, 2, 3\}$ . The matrices of global maps for such homoclinic cycles have the following forms:

$$\begin{pmatrix} * & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \quad \begin{pmatrix} 0 & * & 0 \\ * & 0 & * \\ * & 0 & * \end{pmatrix} \quad \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix} \quad \begin{pmatrix} * & 0 & * \\ * & 0 & * \\ 0 & * & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & * & * \\ 0 & * & * \\ * & 0 & 0 \end{pmatrix} \quad (11)$$

3-2-(1)(23)      3-2-(2)(13)      3-2-(12)(3)      3-2-(13)(2)      3-2-(23)(1)

Poincaré maps for these cycles are listed in table 1. The exponents  $a_j$ ,  $j = 1, 2, 3$ , can be expressed in terms of eigenvalues of the linearisation  $df(\xi)$  by the relations

$$a_1 = c/e, \quad a_2 = -t_1/e \text{ (or } -t/e \text{ if } \dim P^\perp = 1), \quad a_3 = -t_2/e. \quad (12)$$



| Classification    | Poincaré map  | Type  |
|-------------------|---|-------|
| 1 – 1 – (1)       | $g(w) = (c_{11}w^{a_1})$  | A', Z |
| 2 – 1 – (12)      | $g(w, z) = (c_{11}w^{a_1} + c_{12}z w ^{a_2}, c_{21}w^{a_1} + c_{22}z w ^{a_2})$  | A'    |
| 2 – 2 – (1)(2)    | $g(w, z) = (c_{11}w^{a_1}, c_{22}z w ^{a_2})$   | Z     |
| 2 – 2 – (2)(1)    | $g(w, z) = (c_{12}z w ^{a_2}, c_{21}w^{a_1})$   | Z     |
| 3 – 1 – (123)     | $g(w, z_1, z_2) = (c_{11}w^{a_1} + c_{12}z_1 w ^{a_2} + c_{13}z_2 w ^{a_3},$<br>$c_{21}w^{a_1} + c_{22}z_1 w ^{a_2} + c_{23}z_2 w ^{a_3}, c_{31}w^{a_1} + c_{32}z_1 w ^{a_2} + c_{33}z_2 w ^{a_3})$ | A'    |
| 3 – 2 – (12)(3)   | $g(w, z_1, z_2) = (c_{11}w^{a_1} + c_{12}z_1 w ^{a_2}, c_{21}w^{a_1} + c_{22}z_1 w ^{a_2}, c_{33}z_2 w ^{a_3})$   |       |
| 3 – 2 – (1)(23)   | $g(w, z_1, z_2) = (c_{11}w^{a_1}, c_{22}z_1 w ^{a_2} + c_{23}z_2 w ^{a_3}, c_{32}z_1 w ^{a_2} + c_{33}z_2 w ^{a_3})$  |       |
| 3 – 2 – (1)(23)m  | $g(w, u) = (c_{11}w^{a_1} +  w ^{a_2k}\text{Re}((c_{12} + ic_{13})(z_1 + iz_2)^k),$<br>$c_{22}z_1 w ^{a_2} + c_{23}z_1 w ^{a_2}, -c_{23}z_1 w ^{a_2} + c_{22}z_1 w ^{a_2})$                         |       |
| 3 – 2 – (2)(13)   | $g(w, z_1, z_2) = (c_{12}z_1 w ^{a_2}, c_{21}w^{a_1} + c_{23}z_2 w ^{a_3}, c_{31}w^{a_1} + c_{33}z_2 w ^{a_3})$   |       |
| 3 – 2 – (13)(2)   | $g(w, z_1, z_2) = (c_{11}w^{a_1} + c_{13}z_2 w ^{a_3}, c_{21}w^{a_1} + c_{23}z_2 w ^{a_3}, c_{32}z_1 w ^{a_2})$   |       |
| 3 – 2 – (23)(1)   | $g(w, z_1, z_2) = (c_{12}z_1 w ^{a_2} + c_{13}z_2 w ^{a_3}, c_{22}z_1 w ^{a_2} + c_{23}z_2 w ^{a_3}, c_{31}w^{a_1})$  |       |
| 3 – 3 – (1)(2)(3) | $g(w, z_1, z_2) = (c_{11}w^{a_1}, c_{22}z_1 w ^{a_2}, c_{33}z_2 w ^{a_3})$  | Z     |
| 3 – 3 – (1)(3)(2) | $g(w, z_1, z_2) = (c_{11}w^{a_1}, c_{23}z_2 w ^{a_3}, c_{32}z_1 w ^{a_2})$  | Z     |
| 3 – 3 – (2)(1)(3) | $g(w, z_1, z_2) = (c_{12}z_1 w ^{a_2}, c_{21}w^{a_1}, c_{33}z_2 w ^{a_3})$  | Z     |
| 3 – 3 – (2)(3)(1) | $g(w, z_1, z_2) = (c_{12}z_1 w ^{a_2}, c_{23}z_2 w ^{a_3}, c_{31}w^{a_1})$  | Z     |

Table 1: Poincaré maps for different classes of homoclinic cycles in  $\mathbb{R}^5$ . Last column indicates cycles of types A' or Z.  $a_j$  are the ratios (12) of eigenvalues of the linearisation.

## 5 Stability

### 5.1 Stability of cycles of types A and Z

In this subsection we review the results on asymptotic stability of cycles of types A [14, 15] and Z [20].

**Theorem 1** ([14, 15], adapted for homoclinic cycles). *Let  $-c$ ,  $e$  and  $t_j$ ,  $1 \leq j \leq J$ , be the contracting, expanding and transverse eigenvalues of  $df(\xi)$  for the steady states  $\xi$  involved in a homoclinic cycle of type A.*

- (a) *If  $c > e$  and  $t_j < 0$  for all  $1 \leq j \leq J$ , then the cycle is asymptotically stable.*
- (b) *If  $c < e$  or  $t_j > 0$  for some  $j$  then the cycle is completely unstable.*

Stability of the fixed point  $(w, \mathbf{z}) = \mathbf{0}$  of the map  $g$  associated with a type Z homoclinic cycle was studied in [20] by considering the map in the coordinates

$$\boldsymbol{\eta} = (\ln |w|, \ln |z_1|, \dots, \ln |z_{n_t}|), \quad (13)$$

in which the map is linear:

$$g\boldsymbol{\eta} = M\boldsymbol{\eta} + \mathbf{F}. \quad (14)$$

Here

$$M = B \begin{pmatrix} a_1 & 0 & 0 & \dots & 0 \\ a_2 & 1 & 0 & \dots & 0 \\ a_3 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_N & 0 & 0 & \dots & 1 \end{pmatrix} \quad (15)$$

is the transition matrix of the map  $g$ ,  $B$  is a permutation matrix (see section 4), and the entries  $a_j$  in the matrix of the local map are

$$a_1 = c/e \text{ and } a_{j+1} = -t_j/e, \quad 1 \leq j \leq J. \quad (16)$$

Any permutation is a superposition of cyclic permutations. We assume that vectors  $\mathbf{e}_j$  in the basis are ordered in such a way that the first  $n_s$  vectors (the subscript  $s$  stands for significant) are involved in a single cyclic permutation and  $\mathbf{e}_1$  is the contracting eigenvector of  $df(\xi)$ . Matrix  $B$  permutes the vectors:  $\mathbf{e}_1 \rightarrow \mathbf{e}_2 \rightarrow \dots \rightarrow \mathbf{e}_{n_s} \rightarrow \mathbf{e}_1$ . Any eigenvalue of the upper left  $n_s \times n_s$  submatrix of  $M$  is also an eigenvalue of  $M$ , because the upper right  $(J - n_s) \times n_s$  submatrix of  $M$  vanishes. These eigenvalues are called significant; generically they differ from one in absolute value. All other eigenvalues of  $M$  are one in absolute value. Denote by  $\lambda_{\max}$  the largest in absolute value significant eigenvalue of  $M$  and by  $\mathbf{v}^{\max}$  the associated eigenvector. Conditions for asymptotic and fragmentary asymptotic stability of type Z cycles in terms of  $\lambda_{\max}$  and components of  $\mathbf{v}^{\max}$  are stated in theorems 2 and 3:

**Theorem 2** *Let  $M$  be the transition matrix of a homoclinic cycle of type Z. Suppose all transverse eigenvalues of  $df(\xi)$  are negative.*

- (a) *If  $|\lambda_{\max}| > 1$ , then the cycle is asymptotically stable.*
- (b) *If  $|\lambda_{\max}| < 1$ , then the cycle is completely unstable.*

**Theorem 3** *Let  $M$  be the transition matrix of a homoclinic cycle of type Z. The cycle is fragmentarily asymptotically stable if and only if the following conditions are satisfied:*

- (i)  $\lambda_{\max}$  is real;
- (ii)  $\lambda_{\max} > 1$ ;
- (iii)  $v_l^{\max} v_q^{\max} > 0$  for all  $l$  and  $q$ ,  $1 \leq l, q \leq N$ .

## 5.2 Stability of homoclinic cycles in $\mathbb{R}^5$

Conditions for asymptotic stability and fragmentary asymptotic stability for various classes of homoclinic cycles are presented in table 2. For type A' cycles the conditions follow from theorem 1.

For type Z cycles the conditions are determined from theorems 2 and 3 by calculating eigenvalues and eigenvectors of transition matrices. For the 2-2-(·)(·) cycles the transition matrices are

$$\begin{pmatrix} a_1 & 0 \\ a_2 & 1 \end{pmatrix} \quad \begin{pmatrix} a_2 & 1 \\ a_1 & 0 \end{pmatrix} \quad (17)$$

$$2-2-(1)(2) \quad 2-2-(2)(1)$$

The first and second matrices have a one- and two-dimensional significant subspace, respectively. Calculating the eigenvectors and eigenvalues, we determine the conditions for asymptotic stability listed in table 2 (previously found in [16, 20, 21]). When such a cycle is not asymptotically stable, it is completely unstable.

The transition matrices of the 3-3-(·)(·)(·) cycles are

$$\begin{pmatrix} a_1 & 0 & 0 \\ a_2 & 1 & 0 \\ a_3 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} a_1 & 0 & 0 \\ a_3 & 0 & 1 \\ a_2 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} a_2 & 1 & 0 \\ a_1 & 0 & 0 \\ a_3 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} a_2 & 1 & 0 \\ a_3 & 0 & 1 \\ a_1 & 0 & 0 \end{pmatrix} \quad (18)$$

$$3-3-(1)(2)(3) \quad 3-3-(1)(3)(2) \quad 3-3-(2)(1)(3) \quad 3-3-(2)(3)(1)$$

The dimension of the significant subspace of the first, second, third and fourth matrix is one, one, two and three, respectively. For the matrices with one- and two-dimensional significant subspaces, the conditions of theorems 2 and 3 can be expressed in terms of  $a_i$  by explicitly calculating the eigenvectors; the respective maps are either asymptotically stable or completely unstable. For the fourth matrix, the relations between the entries  $a_i$ , that are equivalent to the conditions of the theorems, are derived in appendix B. Substituting (16), one obtains the conditions listed in table 2.

Stability of cycles which are not of types A' or Z was not yet studied. In appendix C we derive conditions for asymptotic and fragmentary asymptotic stability of the 3-2-(2)(13) cycle. Its Poincaré map is

$$g(w, z_1, z_2) = (c_{12}z_1|w|^{a_2}, c_{21}w^{a_1} + c_{23}z_2|w|^{a_3}, c_{31}w^{a_1} + c_{33}z_2|w|^{a_3}).$$

If  $w$ ,  $z_1$  and  $z_2$  are small, then in the domains  $w^{a_1-a_3} \gg z_2$  and  $w^{a_1-a_3} \ll z_2$  (which we denote by  $\Omega_I$  and  $\Omega_{II}$ , respectively), the map  $g$  is approximated by the maps

$$g_I(w, z_1, z_2) = (c_{12}z_1|w|^{a_2}, c_{21}w^{a_1}, c_{31}w^{a_1}) \text{ and } g_{II}(w, z_1, z_2) = (c_{12}z_1|w|^{a_2}, c_{23}z_2|w|^{a_3}, c_{33}z_2|w|^{a_3}),$$

respectively. Let  $M_I$  and  $M_{II}$  denote the transition matrices of the two maps,

$$M_I = \begin{pmatrix} a_2 & 1 & 0 \\ a_1 & 0 & 0 \\ a_1 & 0 & 0 \end{pmatrix}, \quad M_{II} = \begin{pmatrix} a_2 & 1 & 0 \\ a_3 & 0 & 1 \\ a_3 & 0 & 1 \end{pmatrix},$$

$\lambda^I$  and  $\lambda^{II}$  their largest in absolute value eigenvalues,  $\mathbf{v}^I$  and  $\mathbf{v}^{II}$  the respective associated eigenvectors,

$$\mathbf{v}^I = (1, \lambda^I - a_2, \lambda^I - a_2) \text{ and } \mathbf{v}^{II} = (1, \lambda^{II} - a_2, \lambda^{II} - a_2),$$

and  $\lambda_{2,3}^I$  and  $\lambda_{2,3}^{II}$  the remaining eigenvalues.

Theorems 4 and 5 (see appendix C) can be summarised as follows:

- (i) the cycle is asymptotically stable if and only if all entries of the transition matrices  $M_I$  and  $M_{II}$  are non-negative and  $\min(|\lambda^I|, |\lambda^{II}|) > 1$ ;
- (ii) the cycle is fragmentarily asymptotically stable if and only if at least one of the following conditions is satisfied:
  - the matrix  $M_I$  satisfies the conditions of theorem 3 (namely,  $\lambda^I$  is real,  $\lambda^I > 1$ ,  $\lambda^I > |\min(\lambda_2^I, \lambda_3^I)|$  and  $v_k^I v_l^I > 0$  for all  $1 \leq k, l \leq 3$ ) and  $\lambda^I - a_2 > (a_1 - a_3)^{-1}$ ;
  - the matrix  $M_{II}$  satisfies the conditions of theorem 3 and  $\lambda^{II} - a_2 < (a_1 - a_3)^{-1}$ ;
- (iii) if the cycle is not fragmentarily asymptotically stable, then it is completely unstable.

We use the  $\boldsymbol{\eta}$ -coordinates (13) to define the set

$$\tilde{\Omega}_{I,\epsilon} = \{(\eta_1, \eta_2, \eta_3) : \eta_3/\eta_1 > a_1 - a_3 + \epsilon\}.$$

For any  $\epsilon > 0$  the set  $\Omega_{I,\epsilon}$  is a subset of  $\Omega_I$  (upon the change of coordinates (13)). Note that the condition  $\lambda^I - a_2 > (a_1 - a_3)^{-1}$  for fragmentary asymptotic stability (see (ii)) is equivalent to the condition that the eigenvector  $\mathbf{v}^I$  belongs to  $\tilde{\Omega}_{I,\epsilon}$  for some  $\epsilon > 0$ . An analogous equivalence holds for the matrix  $M_{II}$  and the set

$$\tilde{\Omega}_{II,\epsilon} = \{(\eta_1, \eta_2, \eta_3) : \eta_3/\eta_1 < a_1 - a_3 - \epsilon\}.$$

Conditions (i) and (ii) expressed in terms of eigenvalues of  $df(\xi)$  are listed in table 2. For other cycles from the 3-2-... classes, the results can be obtained similarly — by introducing analogous matrices  $M_I$  and  $M_{II}$  and applying to them the results of theorems 2 and 3 together with the conditions that  $\mathbf{v}^I \subset \tilde{\Omega}_{I,\epsilon}$  and  $\mathbf{v}^{II} \subset \tilde{\Omega}_{II,\epsilon}$  for some  $\epsilon > 0$ .

## 6 Conclusion

We have introduced a classification of simple homoclinic cycles in  $\mathbb{R}^n$  and identified all classes of homoclinic cycles in  $\mathbb{R}^5$ . For each class in  $\mathbb{R}^5$  we have derived necessary and sufficient conditions for asymptotic and fragmentary asymptotic stability in terms of eigenvalues of linearisation near the steady state involved in the cycle. Transition matrices were used in the derivation of conditions for stability for cycles from the new classes, as in [20] for type Z cycles, although now the conditions for stability and the derivations are more involved. The approach based on the use of transition matrices is likely to be fruitful for investigation of stability of other simple cycles, i.e., for homoclinic cycles in  $\mathbb{R}^n$  for  $n > 5$  and for heteroclinic cycles in  $\mathbb{R}^n$  for  $n > 4$ .

In  $\mathbb{R}^5$ , our classification is sufficient to determine the stability of a homoclinic cycle: the stability is controlled, in a class-dependent way, by eigenvalues of the linearisation. It is

| Classification      | Conditions for stability   |
|---------------------|--|
| 1 – 1 – (1)         | A. s.: $c > e$   |
| 2 – 1 – (12)        | A. s.: $c > e, t < 0$  |
| 2 – 2 – (1)(2)      | A. s.: $c > e, t < 0$  |
| 2 – 2 – (2)(1)      | A. s.: $c - t > e, t < 0$  |
| 3 – 1 – (123)       | A. s.: $c > e, t_1 < 0, t_2 < 0$   |
| 3 – 2 – (12)(3)     | A. s.: $c > e, t_1 < 0, t_2 < 0$   |
| 3 – 2 – (1)(23),    | A. s.: $c > e, t_1 < 0, t_2 < 0$   |
| 3 – 2 – (1)(23) $m$ | A. s.: $c > e, t_1 < 0, t_2 < 0$   |
| 3 – 2 – (2)(13)     | A. s.: $c - t_1 > e, t_1 < 0, t_2 < 0$<br>F. a. s.: $t_1(c - t_2) + (c - t_2)^2 - ce > 0, t_1 < e, t_2 < 0$  |
| 3 – 2 – (13)(2)     | A. s.: $c > e, t_1 < 0, t_2 < 0$<br>F. a. s.: $c(c - t_2) - e^2 - t_1e < 0, t_1 < e, t_2 < 0, t_1 + t_2 < 0$<br>or $c(c - t_2) - e^2 - t_1e > 0, t_1 < e, c > e$ |
| 3 – 2 – (23)(1)     | A. s.: $c - \max(t_1, t_2) > e, t_1 < 0, t_2 < 0$  |
| 3 – 3 – (1)(2)(3)   | A. s.: $c > e, t_1 < 0, t_2 < 0$   |
| 3 – 3 – (1)(3)(2)   | A. s.: $c > e, t_1 < 0, t_2 < 0$   |
| 3 – 3 – (2)(1)(3)   | A. s.: $c - t_1 > e, t_1 < 0, t_2 < 0$   |
| 3 – 3 – (2)(3)(1)   | A. s.: $c - t_1 - t_2 > e, t_1 < 0, t_2 < 0$<br>F. a. s.: $c - t_1 - t_2 > e, t_1 t_2 + ce > 0, ct_1^3 + et_2^3 < 0$   |

Table 2: Conditions for asymptotic stability and fragmentary asymptotic stability of different classes of homoclinic cycles in  $\mathbb{R}^5$  in terms of eigenvalues of the linearisation  $df(\xi)$ .

unclear whether the classification suffices to determine conditions for stability in higher-dimensional systems. Note that a finer classification involving structure angles was proposed for homoclinic cycles in  $\mathbb{R}^4$  and  $\mathbb{R}^5$  [25, 26].

A study of stability is often followed by a study of bifurcations. A natural continuation of the present paper is an investigation of resonance bifurcations of simple homoclinic cycles in  $\mathbb{R}^5$ , similar to the investigation [7] for homoclinic cycles in  $\mathbb{R}^4$ .

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## A Global map for the 3-2-(1)(23) homoclinic cycles

Consider a map  $\psi : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  equivariant under a symmetry group  $\Sigma \subset \mathbf{O}(3)$ . Suppose

- (a)  $\Sigma$  decomposes  $\mathbb{R}^3$  into two isotypic components

$$\mathbb{R}^3 = U_1 \oplus U_2,$$

where the dimension of  $U_1$  is one and of  $U_2$  is two;

- (b) for any  $\mathbf{x} \in \mathbb{R}^3$  there exists  $\sigma \in \Sigma$  such that  $\sigma \mathbf{x} \neq \mathbf{x}$ .

Consider the  $(x, z)$  coordinates in  $\mathbb{R}^3$ , where  $x$  is the coordinate in  $U_1$  and  $z$  in  $U_2$ . In this appendix we determine the leading terms of the expansion of  $\psi$  in small  $x$  and  $z$ . We use the following lemma [20]:

**Lemma 2** *Let a group  $\Sigma$  act on a linear space  $V$ . Consider the isotypic decomposition of  $V$  under the action of  $\Sigma$ :*

$$V = U_0 \oplus U_1 \oplus \dots \oplus U_K.$$

*Suppose*

- *the action of  $\Sigma$  on  $U_0$  is trivial;*
- *any  $\sigma \in \Sigma$  acts on a  $U_k$ ,  $1 \leq k \leq K$ , either as  $I$  or as  $-I$ .*

*Then for any collection of subscripts  $1 \leq i_1, \dots, i_l \leq K$  there exists a subgroup  $G_{i_1, \dots, i_l} \subset \Sigma$  such that the subspace*

$$V_{i_1, \dots, i_l} = U_0 \oplus U_{i_1} \oplus \dots \oplus U_{i_l}$$

*is a fixed point subspace of the group  $G_{i_1, \dots, i_l}$ .*

By (a) and (b), there exists a symmetry  $\sigma_1 \in \Sigma$  such that  $\sigma_1(x, 0) = (-x, 0)$ .  $\Sigma$  can act on  $U_2$  in three ways:

- (i) there exists  $\sigma_2 \in \Sigma$  such that  $\sigma_2 \neq \sigma_1^k$  for any  $k$  and  $\sigma_2(0, z) = (0, -z)$ ;
- (ii) no  $\sigma_2$  satisfying (i) exists; there exists  $\sigma_3 \in \Sigma$  such that  $\sigma_3(0, z) = (0, e^{2\pi i/k} z)$ , where  $k > 1$  is odd, and  $\sigma_1(x, z) = (-x, z)$ ;
- (iii) no  $\sigma_2$  satisfying (i) exists; there exist  $\sigma_3 \in \Sigma$  such that  $\sigma_3(0, z) = (0, e^{2\pi i/k} z)$ , where  $k > 1$  is odd, and  $\sigma_1(x, z) = (-x, -z)$ .

Lemma 1 implies that in case (i) the  $x$ - and  $z$ -components of  $\psi$  are

$$\psi^x = xF(x, z, \bar{z}), \quad \psi^z = zG(x, z, \bar{z}) + \bar{z}^s H(x, z, \bar{z}),$$

where  $F$  is real, generically  $F(0, 0, 0) \neq 0$  and  $G(0, 0, 0) \neq 0$ ,  $s > 0$  is odd (it is determined by  $\Sigma$ ). In cases (ii) and (iii) the components of  $\psi$  can be calculated by simple algebra:

$$\begin{aligned} \text{(ii)} \quad & \psi^x = xF(x, z, \bar{z}), \quad \psi^z = zG(x, z, \bar{z}) + \bar{z}^{k-1} H(x, z, \bar{z}) \\ \text{(iii)} \quad & \psi^x = xF(x, z, \bar{z}) + z^k J(x, z, \bar{z}) + \bar{z}^k \bar{J}(x, z, \bar{z}), \quad \psi^z = zG(x, z, \bar{z}) \end{aligned}$$

where  $F$  is real, generically  $F(0, 0, 0) \neq 0$ ,  $G(0, 0, 0) \neq 0$  and  $J(0, 0, 0) \neq 0$ . Thus in cases (i) and (ii), for small  $x$  and  $z$  the asymptotically largest terms of  $\psi$  are

$$\psi^x = ax, \quad \psi^z = bz + c\bar{z}^s, \tag{19}$$

where  $a$  is real,  $b$  and  $c$  are complex, and generically  $a \neq 0$  and  $b \neq 0$ . (Here  $b$  and  $c$  are further restricted by the action of other symmetries from  $\Sigma$ , but these restrictions are



insignificant for the study of stability of the Poincaré map.) In case (iii) the asymptotically largest terms of the map  $\psi$  are

$$\psi^x = ax + bz^k + \bar{b}\bar{z}^k, \quad \psi^z = cz, \quad (20)$$

where  $a$  is real,  $b$  and  $c$  are complex, and generically  $a \neq 0$ ,  $b \neq 0$  and  $c \neq 0$ . We refer to a cycle with the global map (19) in cases (i) or (ii) as a 3-2-(2)(13) cycle, and to a cycle with the global map (20) in case (iii) as a 3-2-(2)(13)m cycle.

## B Stability of the 3-3-(2)(1)(3) homoclinic cycle

In this appendix we derive necessary and sufficient conditions for asymptotic stability and fragmentary asymptotic stability of the 3-3-(2)(1)(3) homoclinic cycle in terms of eigenvalues of the linearisation near the equilibrium. As noted in section 4, such a cycle is of type Z. Conditions for stability of type Z cycles in terms of eigenvalues and eigenvectors of their transition matrices are given by theorems 2 and 3. The transition matrix of the cycle is

$$M = \begin{pmatrix} a_2 & 1 & 0 \\ a_3 & 0 & 1 \\ a_1 & 0 & 0 \end{pmatrix} \quad (21)$$

(see (18)), where  $a_i$  are related to eigenvalues of the linearisation by (16),  $a_1 > 0$  ( $a_2$  and  $a_3$  can have arbitrary signs).

Let  $\lambda_1$ ,  $\lambda_2$  and  $\lambda_3$  be the eigenvalues of (21);  $\lambda_1$  denotes the largest eigenvalue if all eigenvalues are real, or the real eigenvalue if the matrix has complex ones. Let  $\mathbf{w}$  denote the eigenvector associated with  $\lambda_1$ . By theorem 2, the necessary and sufficient conditions for asymptotic stability are

$$a_2 > 0, \quad a_3 > 0, \quad \max_j |\lambda_j| > 1. \quad (22)$$

By theorem 3, the cycle is fragmentarily asymptotically stable if and only if

$$\lambda_1 > 1; \quad (23)$$

$$\lambda_1 > \max(|\lambda_2|, |\lambda_3|); \quad (24)$$

$$w_i w_j > 0 \text{ for any } 1 \leq i, j \leq 3. \quad (25)$$

By applying the following lemmas, one can avoid calculating the eigenvalues  $\lambda_i$  by Cardano's formulae for the roots of a cubic polynomial.

**Lemma 3** *Let all  $a_i > 0$  in matrix (21). Then  $\max_j |\lambda_j| > 1$  if and only if*

$$a_1 + a_2 + a_3 > 1. \quad (26)$$

**Proof:** Eigenvalues of matrix (21) are roots of its characteristic polynomial

$$p_M(\lambda) = -\lambda^3 + a_2\lambda^2 + a_3\lambda + a_1. \quad (27)$$

Suppose the inequality (26) is satisfied. This implies  $p_M(1) > 0$ . Since  $p_M(\infty) < 0$ , the polynomial  $p_M$  has a root larger than one, and thus  $\max_j |\lambda_j| > 1$ .

We prove now the converse. Denote by  $\lambda_{\max}$  the maximal in absolute value root of  $p_M(\lambda)$ . Since all  $a_i > 0$  and  $|\lambda_{\max}| > 1$ ,

$$a_1 + a_2 + a_3 > a_2 + \frac{a_3}{|\lambda_{\max}|} + \frac{a_1}{|\lambda_{\max}|^2} \geq \left| a_2 + \frac{a_3}{\lambda_{\max}} + \frac{a_1}{\lambda_{\max}^2} \right| = |\lambda_{\max}| > 1.$$

**QED**

Components of the eigenvector  $\mathbf{w}$  associated with the eigenvalue  $\lambda_1$  satisfy the equations

$$a_2 w_1 + w_2 = \lambda_1 w_1, \quad (28)$$

$$a_3 w_1 + w_3 = \lambda_1 w_2, \quad (29)$$

$$a_1 w_1 = \lambda_1 w_3. \quad (30)$$

By the Viète formulae for the roots of the characteristic polynomial  $p_M$ ,

$$\lambda_1 + \lambda_2 + \lambda_3 = a_2, \quad (31)$$

$$-\lambda_1 \lambda_2 - \lambda_2 \lambda_3 - \lambda_1 \lambda_3 = a_3, \quad (32)$$

$$\lambda_1 \lambda_2 \lambda_3 = a_1. \quad (33)$$

**Lemma 4** *Eigenvalues and eigenvectors of matrix (21) satisfy conditions (23)-(25) if and only if the following four inequalities hold true:*

$$\begin{aligned} a_1 &> 0, \\ a_1 + a_2 + a_3 &> 1, \\ a_2 a_3 + a_1 &> 0, \\ a_1 a_2^3 + a_3^3 &> 0. \end{aligned} \quad (34)$$

**Proof:** The cubic polynomial  $p_M$  has either three real roots or one real root and two complex conjugate ones. We consider the two cases separately.

Suppose all eigenvalues  $\lambda_i$  are real.

Assume that (23)-(25) hold true. By virtue of (30) and (23), and since  $w_1$  and  $w_3$  have same signs (25), we have  $a_1 > 0$ ; hence by (33)  $\lambda_2$  and  $\lambda_3$  have same signs. Equations (31) and (28) yield

$$-w_1(\lambda_2 + \lambda_3) = w_2, \quad (35)$$

whereby  $\lambda_2 + \lambda_3 < 0$  (see (25)). Therefore,  $\lambda_2 < 0$  and  $\lambda_3 < 0$ . Thus, (24) and the identity

$$(\lambda_1^2 - \lambda_2 \lambda_3)(\lambda_2^2 - \lambda_3 \lambda_1)(\lambda_3^2 - \lambda_1 \lambda_2) = a_1 a_2^3 + a_3^3 \quad (36)$$

(which follows from the Viète formulae) yield

$$a_1 a_2^3 + a_3^3 > 0.$$

The characteristic polynomial  $p_M$  (27) has only one positive root that is larger than one. Since  $p_M(\infty) < 0$ , this implies  $p_M(1) = -1 + a_1 + a_2 + a_3 > 0$ . The identity

$$(\lambda_1 + \lambda_2)(\lambda_2 + \lambda_3)(\lambda_3 + \lambda_1) = -a_2 a_3 - a_1, \quad (37)$$

and (24) yield  $a_2 a_3 + a_1 > 0$ . Thus, all inequalities in (34) are proven.

We prove now the converse assuming that (34) is satisfied. Since  $a_1 > 0$ , by virtue of (33)  $\lambda_1 > 0$  for our ordering of  $\lambda_i$ , and  $\lambda_2 \lambda_3 > 0$ . The inequality  $a_2 a_3 + a_1 > 0$  and (37) imply that  $\lambda_2 < 0$  and  $\lambda_3 < 0$ . From (30) we deduce that  $w_1$  and  $w_3$  have same signs; by virtue of (35)  $w_1$  and  $w_2$  also have same signs, which proves (25). Since  $p_M(1) > 0$  and  $p_M(\infty) < 0$ , (23) holds true. Due to (36) and (37), the inequalities  $a_2 a_3 + a_1 > 0$  and  $a_1 a_2^3 + a_3^3 > 0$  imply that the condition (24) holds true.

Suppose now the polynomial  $p_M$  has one real root  $\lambda_1$  and two complex conjugate roots  $\lambda_{2,3} = \alpha \pm i\beta$ . Identities (37) and (36) yield, respectively,

$$a_2 a_3 + a_1 = -2\alpha((\lambda_1 + \alpha)^2 + \beta^2) \quad (38)$$

and

$$a_1 a_2^3 + a_3^3 = \Theta(\lambda_1^2 - \lambda_2 \lambda_3)(\alpha^2 + \beta^2)^{-1} \quad (39)$$

where

$$\Theta \equiv (\lambda_1(\alpha^2 + \beta^2) - \alpha^3 + 3\alpha\beta^2)^2 + (3\alpha^2\beta - \beta^3)^2 > 0, \quad (40)$$

unless  $a_1 + a_2 + a_3 = 0$  and  $3\alpha^2 = \beta^2$ .

By (39) and (40), condition (24) is equivalent to the inequality  $a_1 a_2^3 + a_3^3 > 0$ . Since  $\lambda_2 \lambda_3 = |\lambda_2|^2$ , (30) and (33) imply that  $w_1$  and  $w_3$  have same signs. By (38) the inequalities  $a_2 a_3 + a_1 > 0$  and  $\alpha < 0$  are equivalent. By virtue of (31), (28) reduces to

$$-2\alpha w_1 = w_2,$$

and hence  $a_2 a_3 + a_1 > 0$  is equivalent to (25).

For the characteristic polynomial  $p_M$  (27) with only one real eigenvalue the conditions (23) and  $a_1 + a_2 + a_3 > 1$  are equivalent. Finally, (33) imply that  $a_1$  and  $\lambda_1$  have same signs. **QED**

Substituting expressions (16) into the inequalities in the statements of lemmas 3 and 4, we establish conditions for stability of the cycle presented in table 2.

## C Stability of the 3-2-(2)(13) homoclinic cycle

In this appendix we derive necessary and sufficient conditions for asymptotic and fragmentary asymptotic stability of the 3-2-(2)(13) homoclinic cycle. The Poincaré map near the cycle is

$$g(w, z_1, z_2) = (Az_1|w|^{a_2}, Bw^{a_1} + Cz_2|w|^{a_3}, Dw^{a_1} + Ez_2|w|^{a_3}) \quad (41)$$

(see table 1). Recall that  $a_1 = -c/e$ , and therefore by the definition of expanding and contracting eigenvalues (see subsection 2.2)  $a_1$  is always positive.

If  $w$ ,  $z_1$  and  $z_2$  are small and  $z_2 \ll w^{a_1-a_3}$ , then the map (41) can be approximated by

$$g_I(w, z_1, z_2) = (z_1|w|^{a_2}, w^{a_1}, w^{a_1}). \quad (42)$$

If  $w$ ,  $z_1$  and  $z_2$  are small and  $z_2 \gg w^{a_1-a_3}$  (which is possible only for  $a_1 > a_3$ ), the map can be approximated by

$$g_{II}(w, z_1, z_2) = (z_1|w|^{a_2}, z_2|w|^{a_3}, z_2|w|^{a_3}). \quad (43)$$

Consider the maps  $g_I$  and  $g_{II}$  restricted to  $\mathbb{R}_+^3$ . In the new coordinates

$$\boldsymbol{\eta} = (\ln w, \ln z_1, \ln z_2), \quad (44)$$

the maps  $g_I$  and  $g_{II}$  are linear. The matrices of the two maps,

$$M_I = \begin{pmatrix} a_2 & 1 & 0 \\ a_1 & 0 & 0 \\ a_1 & 0 & 0 \end{pmatrix} \quad (45)$$

and

$$M_{II} = \begin{pmatrix} a_2 & 1 & 0 \\ a_3 & 0 & 1 \\ a_3 & 0 & 1 \end{pmatrix}, \quad (46)$$

have eigenvalues

$$\lambda_{1,2}^I = \frac{1}{2}(a_2 \pm (a_2^2 + 4a_1)^{1/2}) \text{ and } \lambda_3^I = 0, \quad (47)$$

and

$$\lambda_{1,2}^{II} = \frac{1}{2}(1 + a_2 \pm ((1 + a_2)^2 + 4(a_3 - a_2))^{1/2}) \text{ and } \lambda_3^{II} = 0, \quad (48)$$

respectively. Since  $a_1 > 0$ ,  $\lambda_{1,2}^I$  (47) are real. Eigenvalues  $\lambda_{1,2}^{II}$  are real if and only if

$$(1 - a_2)^2 + 4a_3 \geq 0.$$

We assume now that  $(1 - a_2)^2 + 4a_3 \geq 0$  and denote the largest eigenvalues of the matrices  $M_I$  and  $M_{II}$  by  $\lambda^I$  and  $\lambda^{II}$ . The associated eigenvectors are

$$\mathbf{v}^I = (1, \lambda^I - a_2, \lambda^I - a_2) \text{ and } \mathbf{v}^{II} = (1, \lambda^{II} - a_2, \lambda^{II} - a_2), \quad (49)$$

respectively. Simple algebra reveals that the inequality

$$\lambda^I < \lambda^{II}$$

is equivalent to

$$a_2(a_1 - a_3) + (a_1 - a_3)^2 - a_1 < 0. \quad (50)$$

It is straightforward to establish that (50) is not satisfied, if  $(1 - a_2)^2 + 4a_3 < 0$ .

We present here proofs of theorems and lemmas only when (50) is satisfied. (When the converse holds true, the proofs remain similar.) For convenience, without any loss of generality we also assume

$$A, C, E > 0, \quad B, D < 0 \text{ and } -C/B > -E/D. \quad (51)$$

There exists  $p$ ,  $0 < p < 1$ , such that for any  $p'$  and  $p''$  from the interval  $[-p, p]$  eigenvalues of the matrices

$$M_{I,p'} = \begin{pmatrix} a_2 & 1+p' & 0 \\ a_1 & 0 & 0 \\ a_1 & 0 & 0 \end{pmatrix} \text{ and } M_{II,p''} = \begin{pmatrix} a_2 & 1+p'' & 0 \\ a_3 & 0 & 1 \\ a_3 & 0 & 1 \end{pmatrix}, \quad (52)$$

are real, and the largest eigenvalues (which we denote by  $\lambda_{p'}^I$  and  $\lambda_{p''}^{II}$ , respectively) satisfy the inequalities

$$\lambda_{p'}^I < \lambda_{p''}^{II}, \quad (1 - \lambda^I)(1 - \lambda_{p'}^I) > 0 \text{ and } (1 - \lambda^{II})(1 - \lambda_{p''}^{II}) > 0. \quad (53)$$

Let  $s > 0$  satisfy

$$s < \min\left(\frac{a_1(1-p)}{a_1 - a_3}, \frac{a_1 p}{(a_1 - a_3)(1-p)}\right). \quad (54)$$

For  $a_1 > a_3$ , we partition  $\mathbb{R}^3$  into five regions (if  $a_1 < a_3$ , such partitioning is not needed, see the proof of theorem 4):

$$\begin{aligned} \Omega_I &= \{(w, z_1, z_2) : |w|^{a_1-a_3} > -\frac{C}{B}|z_2|(1+|z_2|^s)\} \\ \Omega_{II} &= \{(w, z_1, z_2) : |w|^{a_1-a_3} < -\frac{E}{D}|z_2|(1-|z_2|^s)\} \\ \Omega_{III} &= \{(w, z_1, z_2) : -\frac{C}{B}z_2(1-|z_2|^s) < w^{a_1-a_3} < -\frac{C}{B}z_2(1+|z_2|^s)\} \\ \Omega_{IV} &= \{(w, z_1, z_2) : -\frac{E}{D}z_2(1-|z_2|^s) < w^{a_1-a_3} < -\frac{E}{D}z_2(1+|z_2|^s)\} \\ \Omega_V &= \{(w, z_1, z_2) : -\frac{D}{E}|z_2|(1-|z_2|^s) < w^{a_1-a_3} < -\frac{B}{C}z_2(1+|z_2|^s), (w, z_1, z_2) \notin \Omega_{III} \cup \Omega_{IV}\}. \end{aligned} \quad (55)$$

Define the sets

$$\omega = \{(w, z_1, z_2) : |z_2|^{1+p} < |z_1| < |z_2|^{1-p}\}, \quad (56)$$

and

$$\omega_J = \omega \cap \Omega_J, \quad 1 \leq J \leq V. \quad (57)$$

$\omega_I$  and  $\omega_{II}$  are partitioned further:  $\omega_I = \omega_{I,I} \cup \omega_{I,II}$  and  $\omega_{II} = \omega_{II,I} \cup \omega_{II,II}$ , where

$$\begin{aligned} \omega_{I,I} &= \omega_I \cap g^{-1}\omega_I \text{ and } \omega_{I,II} = \omega_I \setminus \omega_{I,I}; \\ \omega_{II,II} &= \omega_{II} \cap g^{-1}\omega_{II} \text{ and } \omega_{II,I} = \omega_{II} \setminus \omega_{II,II}. \end{aligned} \quad (58)$$

Denote

$$Q(f_1, f_2, h) = \{(w, z_1, z_2) : f_1(z_2) < w < f_2(z_2), z_1 = h(w, z_2)\}, \quad (59)$$

$$q(f, h) = \{(w, z_1, z_2) : w = f(z_2), z_1 = h(z_2)\}, \quad (60)$$

by  $B_\epsilon$  the  $\epsilon$ -neighbourhood of the point  $(w, z_1, z_2) = \mathbf{0}$  and

$$\Omega_J(\epsilon) = \Omega_J \cap B_\epsilon, \quad \omega_J(\epsilon) = \omega_J \cap B_\epsilon, \quad Q(f_1, f_2, h; \epsilon) = Q(f_1, f_2, h) \cap B_\epsilon, \quad q(f, h; \epsilon) = q(f, h) \cap B_\epsilon.$$

Furthermore,

$$f_1(x) \approx f_2(x) \text{ denotes that } f_1(x) - f_2(x) = o(f_1(x)) \text{ when } x \rightarrow 0$$

and

$$f_1(x) \sim f_2(x) \text{ denotes that } f_1(x) \approx F f_2(x) \text{ for a constant } F \neq 0.$$

For  $a_1 > \max(0, a_3)$ , the following properties of the map  $g$  are direct consequences of (41):

- (a)  $g$  is one-to-one, except for the hyperplane  $w = 0$ ;
- (b) if  $q(f, h) \subset \omega_I$  and  $f \sim z_2^\alpha$ , then  $gq(f, h) = q(\tilde{f}, \tilde{h})$  where  $\tilde{f} \sim z_2^{\tilde{\alpha}}$ ,  $\tilde{h} \sim z_2$  and  $(1 - p + a_2\alpha)a_1^{-1}\alpha^{-1} < \tilde{\alpha} < (1 + p + a_2\alpha)a_1^{-1}\alpha^{-1}$ ;
- (c) if  $q(f, h) \subset \omega_{II}$  and  $f \sim z_2^\alpha$ , then  $gq(f, h) = q(\tilde{f}, \tilde{h})$  where  $\tilde{f} \sim z_2^{\tilde{\alpha}}$ ,  $\tilde{h} \sim z_2$  and  $(1 - p + a_2\alpha)(1 + a_3\alpha)^{-1} < \tilde{\alpha} < (1 + p + a_2\alpha)(1 + a_3\alpha)^{-1}$ ;
- (d) if  $q(f, h) \subset \omega_{III}$  and  $f - (CB^{-1}z_2)^{1/(a_1-a_3)} \sim z_2^\alpha$  (note that  $\alpha > s + (a_1 - a_3)^{-1}$ ), then  $gq(f, h) = q(\tilde{f}, \tilde{h})$ , where  $\tilde{f} \sim z_2^{\tilde{\alpha}}$ ,  $\tilde{h} \sim z_2^\beta$ ,  $\beta = (a_1 - 1 + \alpha(a_1 - a_3))a_1^{-1}$  and  $((1 - p)(a_1 - a_3) + a_2)a_1^{-1} < \tilde{\alpha} < ((1 + p)(a_1 - a_3) + a_2)a_1^{-1}$ ;  $g^2q(f, h) = q(\hat{f}, \hat{h})$  where  $\hat{f} \sim z_2^{\hat{\alpha}}$ ,  $\hat{h} \sim z_2$  and

$$\frac{a_1 - 1 + \alpha(a_1 - a_3)}{a_1((1 + p)(a_1 - a_3) + a_2)} + \frac{a_2}{a_1} < \hat{\alpha} < \frac{a_1 - 1 + \alpha(a_1 - a_3)}{a_1((1 - p)(a_1 - a_3) + a_2)} + \frac{a_2}{a_1};$$

(e) if  $q(f, h) \subset \omega_{IV}$  and  $f - (ED^{-1}z_2)^{1/(a_1-a_3)} \sim z_2^\alpha$  (note that  $\alpha > s + (a_1 - a_3)^{-1}$ ), then  $gq(f, h) = q(\tilde{f}, \tilde{h})$  where  $\tilde{f} \sim z_2^{\tilde{\alpha}}$ ,  $\tilde{h} \sim z_2^\beta$ ,  $\beta = a_1(a_1 - 1 + \alpha(a_1 - a_3))^{-1}$  and

$$((1-p)(a_1-a_3)+a_2)(a_1-1+\alpha(a_1-a_3))^{-1} < \tilde{\alpha} < ((1+p)(a_1-a_3)+a_2)(a_1-1+\alpha(a_1-a_3))^{-1};$$

$g^2q(f, h) = q(\hat{f}, \hat{h})$ , where  $\hat{f} \sim z_2^{\hat{\alpha}}$ ,  $\hat{h} \sim z_2$  and

$$\frac{1}{a_1((1+p)(a_1-a_3)+a_2)} + \frac{a_2}{a_1} < \hat{\alpha} < \frac{1}{a_1((1-p)(a_1-a_3)+a_2)} + \frac{a_2}{a_1};$$

(f) if  $q(f, h) \subset \omega_V$ , then  $gq(f, h) = q(\tilde{f}, \tilde{h})$  where  $f \sim z_2^{\tilde{\alpha}}$ ,  $h \sim z_2^\beta$ ,  $1-p < \beta < 1+p$  and

$$((1-p)(a_1-a_3)+a_2)a_1^{-1} < \tilde{\alpha} < ((1+p)(a_1-a_3)+a_2)a_1^{-1};$$

(j) if  $Q(f_1, f_2, h) \subset \omega_I$ ,  $f_1 \sim z_2^\alpha$  and  $f_1 - f_2 \approx Ff_1z_2^\gamma$ , then  $gQ(f_1, f_2, h) = Q(\tilde{f}_1, \tilde{f}_2, \tilde{h})$ , where  $\tilde{f}_1 - \tilde{f}_2 \approx a_2F\tilde{f}_1z_2^{\tilde{\gamma}}$  and  $\tilde{\gamma} = \gamma a_1^{-1}\alpha^{-1}$ ;

(i) if  $Q(f_1, f_2, h) \subset \omega_{II}$ ,  $f_1 \sim z_2^\alpha$  and  $f_1 - f_2 \approx Ff_1z_2^\gamma$ , then  $gQ(f_1, f_2, h) = Q(\tilde{f}_1, \tilde{f}_2, \tilde{h})$ , where  $\tilde{f}_1 - \tilde{f}_2 \approx a_2F\tilde{f}_1z_2^{\tilde{\gamma}}$  and  $\tilde{\gamma} = \gamma(a_1 + a_3\alpha)^{-1}$ ;

By definitions of the sets  $\Omega_J$ , for  $s$  satisfying (54)

$$g(\Omega_I \cup \Omega_{II} \cup \Omega_V) \subset \omega. \quad (61)$$

If  $a_2 > 0$  and (50) holds true, then the listed above properties of the map  $g$  imply that, for a sufficiently small  $\epsilon$ , the following inclusions take place:

$$\begin{aligned} g(\Omega_{III}(\epsilon)) \cap g(\Omega_{IV}) &\subset \omega_I, \quad g(\Omega_{IV}(\epsilon)) \cap g(\Omega_{IV}) \subset \omega_I, \\ g(\omega_{I,I}(\epsilon)) &\subset \omega_{I,I}, \quad g(\omega_V(\epsilon)) \subset \omega_{I,I}, \quad g^2(\omega_{IV}(\epsilon)) \subset \omega_{I,I}; \end{aligned} \quad (62)$$

if  $a_2 < 0$  and (50) holds true, then, for a sufficiently small  $\epsilon$ ,

$$\begin{aligned} g(\Omega_{III}(\epsilon)) \cap g(\Omega_{III}) &\subset \omega_{II}, \quad g(\Omega_{IV}(\epsilon)) \cap g(\Omega_{III}) \subset \omega_{II}; \\ g(\omega_{II,II}(\epsilon)) &\subset \omega_{II,II}, \quad g(\omega_V(\epsilon)) \subset \omega_{II,II}, \quad g^2(\omega_{III}(\epsilon)) \subset \omega_{II,II}. \end{aligned} \quad (63)$$

For a set  $X \subset \mathbb{R}^3$  we denote by  $\mu_{z_1}(X)$  the measure (in  $\mathbb{R}^2$ ) of the projection of the set  $X$  on the plane  $z_1 = 0$  ( $(w, z_1, z_2)$  being cartesian coordinates in  $\mathbb{R}^3$ ). It is easy to show that for  $f_1 \sim z_2^\alpha$ ,  $f_1 - f_2 \approx Ff_1z_2^\gamma$ ,  $\gamma > \alpha$  and a small  $\epsilon$  we have

$$\mu_{z_1}(Q(f_1, f_2, h; \epsilon)) \approx \begin{cases} (\gamma + 1)^{-1}F\epsilon^{\gamma+1} & , \alpha \geq 1, \\ (\gamma + 1)^{-1}F\epsilon^{(\gamma+1)/\alpha} & , \alpha < 1. \end{cases} \quad (64)$$

**Lemma 5** *Let  $Q_0, Q_1, \dots, Q_K$  be the following sets:*

$$Q_k = Q(f_{k1}, f_{k2}, h), \quad f_{k1} \sim f_{k2} \sim z_2^\alpha, \quad 0 \leq k \leq K, \quad f_{01} - f_{02} \approx Bz_2^\beta,$$

$$Q_k \subset Q_0, \quad f_{k1} - f_{k2} \approx C_k z_2^{\gamma_k}, \quad 1 \leq k \leq K,$$

where  $\beta > \alpha$ ,  $\gamma_k > \beta$ ,  $c_k = \gamma_k/\beta$  and  $\beta > \beta_m$ .

For any  $d > 0$  there exists

$$\epsilon = \epsilon(B, C_1, \dots, C_K, \beta_m, c_1, \dots, c_K, d) > 0$$

such that

$$\sum_{k=1}^K \mu_{z_1}(Q_k \cap B_\epsilon) < d \mu_{z_1}(Q_0 \cap B_\epsilon).$$

**Proof:** We present here a proof for  $\alpha \geq 1$  (the proof for  $\alpha < 1$  is similar). By (64),

$$\frac{\mu_{z_1}(Q_k \cap B_\epsilon)}{\mu_{z_1}(Q_0 \cap B_\epsilon)} \approx \frac{C_k(\beta + 1)}{B(\gamma_k + 1)} \epsilon^{\gamma_k - \beta} < \frac{C_k(\beta_m + 1)}{B(c_k \beta_m + 1)} \epsilon^{\beta_m(c_k - 1)}.$$

Any  $\epsilon$ ,  $0 < \epsilon < 1$ , such that

$$\epsilon < \min_{1 \leq k \leq K} \left( \frac{d}{K} \frac{B(c_k \beta_m + 1)}{C_k(\beta_m + 1)} \right)^{1/(\beta_m(c_k - 1))}$$

satisfies all conditions of the lemma. **QED**

**Lemma 6** *Let the map  $r : \mathbb{R} \rightarrow \mathbb{R}$ ,*

$$r(x) = \frac{ax + b}{cx + d},$$

*be defined for all  $x \in [x_1, x_3]$ , where  $x_3 > x_1$ . Denote the superposition of  $k$  (identical) maps  $r$  by*

$$r^k(x) = \underbrace{r \circ \dots \circ r}_{k \text{ times}}(x).$$

(a) *If  $r(x_1) > x_1$ ,  $r(x_3) < x_3$  and  $r(x_2) < x_2$  for some  $x_2$ ,  $x_1 < x_2 < x_3$ , then*

$$x - r(x) \geq \min(x_2 - r(x_2), x_3 - r(x_3)) \text{ for any } x \in [x_2, x_3].$$

*Hence, there exists  $K > 0$  such that*

$$r^k(x) < x_2 \text{ for any } x \in [x_2, x_3] \text{ and for some } 0 < k = k(x) < K.$$

(b) *If  $r(x_1) < x_1$ ,  $r(x_3) > x_3$  and  $r(x_2) > x_2$  for some  $x_2$ ,  $x_1 < x_2 < x_3$ , then*

$$r(x) - x \geq \min(r(x_2) - x_2, r(x_3) - x_3) \text{ for any } x \in [x_2, x_3].$$



Hence, there exists  $K > 0$  such that

$$r^k(x) > x_3 \text{ for any } x \in [x_2, x_3] \text{ and for some } 0 < k = k(x) < K.$$

(c) If  $r(x_1) < x_1$  and  $r(x_3) > x_3$ , then there exists  $x_2$ ,  $x_1 < x_2 < x_3$ , such that  $r(x_2) = x_2$  and

$$\text{for any } x \in [x_1, x_2[ \text{ there exists } k = k(x) > 0 \text{ such that } r^k(x) < x_1,$$

$$\text{for any } x \in ]x_2, x_3] \text{ there exists } k = k(x) > 0 \text{ such that } r^k(x) > x_3.$$

(d) If  $r(x_1) > x_1$  and  $r(x_3) < x_3$ , then

$$r(x) \in [x_1, x_3] \text{ for any } x \in [x_1, x_3];$$

there exists  $x_2$ ,  $x_1 < x_2 < x_3$ , such that

$$r(x_2) = x_2 \text{ and } \lim_{k \rightarrow \infty} r^k(x) = x_2 \text{ for any } x \in [x_1, x_3].$$

**Proof:** We present here a proof for case (a) (the proofs for other cases are similar). The function

$$h(x) = x - r(x) = x - \frac{a}{c} - \frac{b - adc^{-1}}{cx + d}$$

has either no local extrema, or two local extrema  $x_{\text{extr}}$  satisfying

$$(cx_{\text{extr}} + d)^2 + cb - ad = 0.$$

In the former case,  $h$  is a growing function on each interval of its domain, and the statement is trivial. In the latter case, the local extrema are separated by the point  $x = -dc^{-1}$ , where  $r(x)$  is undefined. Hence, any interval, where  $h(x)$  is defined, contains at most one local extremum. If a local extremum belongs to  $[x_2, x_3]$ , the extremum is a local maximum because of the condition  $h(x_2) > h(x_1)$ . Therefore,  $h(x)$  takes the minimum over  $[x_2, x_3]$  at an endpoint of the interval.

Set

$$C = -\min(r(x_2) - x_2, r(x_3) - x_3) > 0 \text{ and } K = [(x_3 - x_2)/C] + 1.$$

(Here  $[q]$  denotes the largest integer not exceeding a real number  $q$ .) For any  $x \in [x_2, x_3]$

$$x_2 - r^K(x) = (x_2 - x) + (x - r(x)) + \dots + (r^{K-1}(x) - r^K(x)) > (x_2 - x) + KC = x_3 - x > 0.$$

**QED**

In the following two lemmas we reformulate (omitting proofs) the results of [20] on asymptotic stability of the fixed point  $\mathbf{x} = \mathbf{0}$  of a map  $\tilde{g} : \mathbb{R}^J \rightarrow \mathbb{R}^J$ ,

$$\tilde{g}(\mathbf{x}) = (|x_1|^{a_{11}}|x_2|^{a_{12}} \dots |x_J|^{a_{1J}}, \dots, |x_1|^{a_{J1}}|x_2|^{a_{J2}} \dots |x_J|^{a_{JJ}}) \quad (65)$$

in terms of eigenvalues and eigenvectors of its transition matrix

$$M = \begin{pmatrix} a_{11} & \dots & a_{1J} \\ \cdot & \dots & \cdot \\ a_{J1} & \dots & a_{JJ} \end{pmatrix}. \quad (66)$$

To prove stability, the coordinates

$$(\eta_1, \dots, \eta_J) = (\ln |x_1|, \dots, \ln |x_J|)$$

were introduced in [20]. In the new coordinates the map  $\tilde{g}$  becomes linear,  $\tilde{g}(\mathbf{x}) = M\boldsymbol{\eta}$ . The initial condition  $\boldsymbol{\eta}$  was decomposed in the basis of eigenvectors of  $M$ , denoted by  $\boldsymbol{\zeta}_j$ :

$$\boldsymbol{\eta} = \sum_{j=1}^J b_j \boldsymbol{\zeta}_j. \quad (67)$$

Let  $\lambda_{\max}$  and  $\boldsymbol{\zeta}^{\max}$  denote the largest in absolute value eigenvalue of  $M$  and the associated eigenvector, respectively. The proof exploited the fact that for  $k \rightarrow \infty$  the iterates  $M^k \boldsymbol{\eta}$  become aligned with  $b_{\max} \lambda_{\max}^k \boldsymbol{\zeta}^{\max}$ , where  $b_{\max}$  is the factor in front of  $\boldsymbol{\zeta}^{\max}$  in the decomposition (67).

**Lemma 7** *Let all eigenvalues of the transition matrix (66) of the map  $\tilde{g}$  (65) differ from one in absolute value and all entries be non-negative.*

- (a) *If  $|\lambda_{\max}| > 1$ , then the fixed point  $\mathbf{x} = \mathbf{0}$  of  $\tilde{g}$  is asymptotically stable, i.e., for any  $\delta > 0$  there exists  $\epsilon > 0$  such that any  $\mathbf{x}$ ,  $|\mathbf{x}| < \epsilon$ , satisfies*

$$|\tilde{g}^k(\mathbf{x})| < \delta \text{ for all } k \geq 0 \text{ and } \lim_{k \rightarrow \infty} \tilde{g}^k(\mathbf{x}) = \mathbf{0}.$$

- (b) *If  $|\lambda_{\max}| < 1$ , then the fixed point  $\mathbf{x} = \mathbf{0}$  of  $\tilde{g}$  is completely unstable, i.e., there exists  $\delta > 0$  such that for any  $\mathbf{x} \neq \mathbf{0}$  there exists  $k \geq 0$  satisfying*

$$|\tilde{g}^k(\mathbf{x})| > \delta.$$

**Lemma 8** *Let the absolute values of all eigenvalues of the transition matrix of the map  $\tilde{g}$  (65) differ from one.*

- (a) *If*

$$\lambda_{\max} \text{ is real, } \lambda_{\max} > 1 \text{ and } \zeta_l^{\max} \zeta_s^{\max} > 0 \text{ for all } 1 \leq l, s \leq J, \quad (68)$$

*then the fixed point  $\mathbf{x} = \mathbf{0}$  of the map  $\tilde{g}$  is fragmentarily asymptotically stable, i.e., for any  $\delta > 0$  the measure of the set of points  $\mathbf{x}$  such that*

$$|\tilde{g}^k(\mathbf{x})| < \delta \text{ for all } k \geq 0 \text{ and } \lim_{k \rightarrow \infty} \tilde{g}^k(\mathbf{x}) = \mathbf{0} \quad (69)$$

*is positive. For a sufficiently large  $C > 0$ , the condition (69) is satisfied by all points in a small neighbourhood of  $-C\boldsymbol{\zeta}^{\max}$ .*

(b) If at least one of the conditions in (68) is not satisfied, then the fixed point  $\mathbf{x} = \mathbf{0}$  of the map  $\tilde{g}$  is completely unstable, i.e., there exist  $\delta > 0$  and, for almost all  $\mathbf{x}$  (for all  $\mathbf{x}$  except for a set of a zero measure),  $k = k(\mathbf{x}) \geq 0$  such that

$$|\tilde{g}^k(\mathbf{x})| > \delta.$$

**Lemma 9** Suppose the inequalities

$$\lambda^I > 1, \lambda^{II} > 1, a_1 > 0, a_3 > a_2 > 0$$

hold true for matrices  $M_I$  (45) and  $M_{II}$  (46). (The first inequality implies that  $a_1 + a_2 > 1$ .) Then the largest eigenvalue  $\lambda^{I,II}$  of the product  $M_I M_{II}$  is real and

$$\lambda^{I,II} > 1.$$

**Proof:** The characteristic equation for the matrix  $M_I M_{II}$  is

$$\lambda^3 - \lambda^2(a_2^2 + a_3 + a_1) - \lambda a_1(a_3 - a_2) = 0.$$

One root is zero and the remaining two satisfy

$$\lambda_1 \lambda_2 = a_1(a_2 - a_3) \text{ and } \lambda_1 + \lambda_2 = a_2^2 + a_3 + a_1.$$

By conditions of the lemma, the product is negative, and the sum is positive and larger than one. Therefore,  $\lambda_1$  and  $\lambda_2$  are real,  $\lambda_2 < 0$  and  $\lambda_1 > \max(1, |\lambda_2|)$ . **QED**

We say that  $\mathbf{x} < \mathbf{y}$ , where  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^J$ , if  $x_j < y_j$  for any  $1 \leq j \leq J$ .

**Definition 14** A map  $g : \mathbb{R}^J \rightarrow \mathbb{R}^J$  is bounded from above (from below) in  $V \subset \mathbb{R}^J$  by a map  $\tilde{g} : \mathbb{R}^J \rightarrow \mathbb{R}^J$ , if  $\tilde{g}(\mathbf{x}) > g(\mathbf{x})$  ( $\tilde{g}(\mathbf{x}) < g(\mathbf{x})$ , respectively) for any  $\mathbf{x} \in V$ .

**Lemma 10** Suppose all entries of the transition matrix  $M$  (66) of the map  $\tilde{g}$  (65) are non-negative. If  $|g|$  is bounded from above (below) by  $\tilde{g}$  in  $V$ , then  $|g^K|$  is bounded from above (below) by  $\tilde{g}^K$  in  $W \subset V$ , where

$$W = \{\mathbf{x} \in V : g^k(\mathbf{x}) \in V \text{ and } |g^k(\mathbf{x})| \in V \text{ for all } 0 \leq k \leq K\}.$$

**Proof:** We present here a proof for  $|g|$  bounded from above by  $\tilde{g}$  (the proof for  $|g|$  bounded from below by  $\tilde{g}$  is similar). Note that if  $|\mathbf{x}| < |\mathbf{y}|$ , then

$$|g(\mathbf{x})| < \tilde{g}(\mathbf{x}) < \tilde{g}(\mathbf{y}).$$

We prove that

$$|g^K(\mathbf{x})| < \tilde{g}^K(\mathbf{x}) \tag{70}$$

by induction in  $K$ . Suppose (70) holds true for  $l = K - 1$ . Then

$$|gg^{K-1}(\mathbf{x})| < \tilde{g}|g^{K-1}(\mathbf{x})| < \tilde{g}^K(\mathbf{x}).$$

**QED**

**Theorem 4** Consider the map

$$g(w, z_1, z_2) = (Az_1|w|^{a_2}, Bw^{a_1} + Cz_2|w|^{a_3}, Dw^{a_1} + Ez_2|w|^{a_3}), \quad (71)$$

where  $a_1 > 0$  and the exponents  $a_1$ ,  $a_2$  and  $a_3$  satisfy the inequality (50).

(a) If

$$a_2 > 0, \quad a_3 > 0 \text{ and } a_1 + a_2 > 1, \quad (72)$$

then the fixed point  $(w, z_1, z_2) = \mathbf{0}$  of the map  $g$  is asymptotically stable.

(b) If

$$a_2 < 0 \text{ or } a_3 < 0 \text{ or } a_1 + a_2 < 1,$$

then the fixed point  $(w, z_1, z_2) = \mathbf{0}$  of the map  $g$  is completely unstable.

**Proof:** Define the maps

$$\begin{aligned} g_{d,I}(w, z_1, z_2) &= (|z_1||w|^{a_2+d}, |w|^{a_1+d}, |w|^{a_1+d}) \text{ and} \\ g_{d,II}(d)(w, z_1, z_2) &= (|z_1||w|^{a_2+d}, |z_2||w|^{a_3+d}, |z_2||w|^{a_3+d}), \end{aligned} \quad (73)$$

whose transition matrices are, respectively,

$$M_{d,I} = \begin{pmatrix} a_2 + d & 1 & 0 \\ a_1 + d & 0 & 0 \\ a_1 + d & 0 & 0 \end{pmatrix} \text{ and } M_{d,II} = \begin{pmatrix} a_2 + d & 1 & 0 \\ a_3 + d & 0 & 1 \\ a_3 + d & 0 & 1 \end{pmatrix}. \quad (74)$$

If  $a_1 < a_3$ , then for small  $w$ ,  $z_1$  and  $z_2$  the map (71) can be approximated by  $g_I$  (i.e., for a sufficiently small  $\epsilon$  and arbitrarily small  $|d|$ , the map  $g$  is bounded in  $B_\epsilon$  from above by  $g_{d,I}$  for  $d < 0$  and bounded from below by  $g_{d,I}$  for  $d > 0$ ). The theorem follows from lemma 10. In what follows, we present the proof for the case  $a_1 > a_3$ .

(a) Recall that (50) and (72) imply that  $\lambda_{II}$  is real and  $\lambda_{II} > \lambda_I > 1$ . Also recall that a fixed point  $(w, z_1, z_2) = \mathbf{0}$  of a map  $g$  is asymptotically stable, if for any  $\delta > 0$  there exists  $\epsilon > 0$  such that

$$|g^k(w, z_1, z_2)| < \delta \text{ for all } k \geq 0 \text{ and } \lim_{k \rightarrow \infty} g^k(w, z_1, z_2) = \mathbf{0} \quad (75)$$

for any  $(w, z_1, z_2) \in B_\epsilon$ .

We show that the iterates  $g^k(w, z_1, z_2)$  are bounded from above by constructing different bounds for the cases  $a_2 < a_3$  and  $a_2 > a_3$ . If  $a_2 < a_3$ , we show that

$$|g^k(w, z_1, z_2)| < g_{d,II}^{k_3}(g_{d,I}g_{d,II})^{k_2}g_{d,I}^{k_1}(w, z_1, z_2) \text{ where } k_1 + 2k_2 + k_3 = k \quad (76)$$

for any  $(w, z_1, z_2) \in B_{\epsilon(d)}$ , any  $d < 0$  and a sufficiently small  $\epsilon(d) > 0$ . Since for a small  $d$  transition matrices of the maps  $g_{d,II}$ ,  $g_{d,I}$  and  $g_{d,I}g_{d,II}$  satisfy conditions of lemma 7, this bound implies (75). If  $a_2 > a_3$  (the proof for this case is not presented), then

$$|g^k(w, z_1, z_2)| < g_{d,I}^{k_3} g_{d,II}^{k_2} g_{d,I}^{k_1}(w, z_1, z_2) \text{ where } k_1 + k_2 + k_3 = k$$

for any  $(w, z_1, z_2) \in B_{\epsilon(d)}$ , any  $d < 0$  and a sufficiently small  $\epsilon(d) > 0$ . To prove (76), we examine the location of the sets  $g\Omega_J$  and  $g\omega_J$ .

- We prove (75) for  $(w, z_1, z_2) \in \omega_{I,I}$ . In  $\omega_{I,I}(\epsilon)$  for a sufficiently small  $\epsilon$ ,  $g(w, z_1, z_2) \in \omega_{I,I}$  (see (62)) and the map  $g$  is bounded from above by  $g_{d,I}$  for  $d < 0$ . Hence, by lemma 10 (applicable by virtue of (72)), for a sufficiently small  $d$  the map  $g^k$  is bounded from above in  $\omega_{I,I}(\epsilon)$  by  $g_{d,I}^k$ . By (72), the map  $g_I$  (and hence  $g_{d,I}$  for a sufficiently small  $d$ ) satisfies the conditions of lemma 7 for asymptotic stability of  $(w, z_1, z_2) = \mathbf{0}$ . By this lemma applied to  $g_{d,I}$ , for any  $\delta > 0$  there exists  $\epsilon_I > 0$  such that

$$\begin{aligned} & \text{for any } (w, z_1, z_2) \in \omega_{I,I}(\epsilon_I), \\ & |g^k(w, z_1, z_2)| < g_{d,I}^k(w, z_1, z_2) < \delta \text{ for any } k \geq 0 \text{ and } \lim_{k \rightarrow \infty} g^k(w, z_1, z_2) = \mathbf{0}. \end{aligned} \quad (77)$$

- Now we prove (75) for any  $(w, z_1, z_2) \in \omega$ .

Depending on the values of  $a_1$ ,  $a_2$  and  $a_3$ , there are three possibilities for the location of  $g(\omega_{II}(\epsilon))$ :

- (i)  $g(\omega_{II}(\epsilon)) \subset \omega_{I,I}$  for some  $\epsilon > 0$ ;
- (ii)  $g(\omega_{II}(\epsilon)) \not\subset \omega_{I,I}$  for any  $\epsilon > 0$  and  $(g\omega_{II}(\delta)) \cap \omega_{II} \neq \emptyset$  for any  $\delta > 0$  ( $a_2 > a_3$  is required for this case);
- (iii)  $g(\omega_{II}(\epsilon)) \not\subset \omega_{I,I}$  for any  $\epsilon > 0$ ; there exists  $\delta_0 > 0$  such that  $g(\omega_{II}(\delta)) \cap \omega_{II} = \emptyset$  for all  $\delta > \delta_0$  ( $a_2 < a_3$  is required for this case).

Proofs for these cases are similar, although differ in details. Only a proof for case (iii) is presented here.

By definition of sets  $\omega_J$  and properties (b)-(f) of  $g$ , when  $\epsilon$  is sufficiently small we have in case (iii):

$$\begin{aligned} g(\omega_{II}(\epsilon)) &\subset \omega_I, & g(\omega_{III}(\epsilon)) &\subset \Omega_I, & g(\omega_{IV}(\epsilon)) &\subset \Omega_I, & g(\omega_V(\epsilon)) &\subset \omega_{I,I}, \\ g(\omega_{I,II}(\epsilon)) &\subset \omega \setminus \omega_{I,II}, & g^2(\omega_{III}(\epsilon)) &\subset \omega \setminus \omega_{I,II}, & g^2(\omega_{IV}(\epsilon)) &\subset \omega_{I,I}. \end{aligned}$$

For a small enough  $\epsilon$ , the map  $g$  is bounded from above by  $g_{d,I}$  in  $\Omega_I(\epsilon)$  and by  $g_{d,II}$  in  $\omega(\epsilon) \setminus \omega_I$ . Hence, if  $(w, z_1, z_2) \in \omega(\epsilon) \setminus \omega_I$ , then the iterates  $g^k(w, z_1, z_2)$  are bounded from above:

$$|g^k(w, z_1, z_2)| < \begin{cases} g_{d,II}(g_{d,I}g_{d,II})^{(k-1)/2}(w, z_1, z_2) & \text{for odd } k, \\ (g_{d,I}g_{d,II})^{k/2}(w, z_1, z_2) & \text{for even } k \end{cases}$$

till  $g^k(w, z_1, z_2)$  gets into  $\omega_{I,I}$  (if this happens). If  $(w, z_1, z_2) \in \omega_{I,II}(\epsilon)$ , then the bounds

$$|g^k(w, z_1, z_2)| < \begin{cases} (g_{d,I}g_{d,II})^{(k-1)/2}g_{d,I}(w, z_1, z_2) & \text{for odd } k, \\ g_{d,II}(g_{d,II}g_{d,I})^{(k-2)/2}g_{d,I}(w, z_1, z_2) & \text{for even } k \end{cases}$$

hold true while  $g^k(w, z_1, z_2) \notin \omega_{I,I}$ . Thus by lemmas 7, 9 and 10, for  $\epsilon_I$  defined in (77) there exists  $\epsilon_{II} > 0$  such that

$$|g^k(w, z_1, z_2)| < \epsilon_I \quad (78)$$

for any  $(w, z_1, z_2) \in \omega(\epsilon_{II}) \setminus \omega_{I,I}$  while  $g^k(w, z_1, z_2) \notin \omega_{I,I}$ . If  $g^k(w, z_1, z_2) \notin \omega_{I,I}$  for any  $k > 0$ , then

$$\lim_{k \rightarrow \infty} g^k(w, z_1, z_2) = 0. \quad (79)$$

• Finally, we prove (75) for  $(w, z_1, z_2) \notin \omega$ . For a sufficiently small  $\epsilon > 0$ , the map  $g$  is bounded from above by  $g_{d,I}$  in  $\Omega_I(\epsilon)$  and by  $g_{d,II}$  in  $B_\epsilon \setminus \Omega_I$ . Hence, by (61), if  $(w, z_1, z_2) \in B_\epsilon \setminus \omega$ , then either

$$|g(w, z_1, z_2)| < g_{d,I}(w, z_1, z_2) \text{ and } g(w, z_1, z_2) \in \omega$$

or

$$|g^k(w, z_1, z_2)| < g_{d,II}^k(w, z_1, z_2) \text{ while } g^{k-1}(w, z_1, z_2) \notin \omega.$$

Thus, there exists  $\epsilon_{III} > 0$  such that for any  $(w, z_1, z_2) \in B_{\epsilon_{III}}$

$$|g^k(w, z_1, z_2)| < \epsilon_{II} \text{ while } g^{k-1}(w, z_1, z_2) \notin \omega. \quad (80)$$

If  $g^k(w, z_1, z_2) \notin \omega$  for any  $k > 0$ , then

$$\lim_{k \rightarrow \infty} g^k(w, z_1, z_2) = 0. \quad (81)$$

Together bounds (77), (78) and (80) imply that for any  $(w, z_1, z_2) \in B_{\epsilon_{III}}$

$$|g^k(w, z_1, z_2)| < \delta \text{ for any } k \leq 0,$$

and relations (77), (79) and (81) imply

$$\lim_{k \rightarrow \infty} g^k(w, z_1, z_2) = 0.$$

Part (a) is proven.

(b.1) Let us consider the case

$$a_2 > 0, \quad a_3 > 0, \quad a_1 + a_2 < 1.$$

(The last inequality is equivalent to  $\lambda^I < 1$ .) We will show that there exists  $\delta > 0$  such that for almost all  $(w, z_1, z_2)$

$$|g^k(w, z_1, z_2)| > \delta \text{ for some } k \geq 0. \quad (82)$$

• We prove (82) for  $(w, z_1, z_2) \in \omega_{I,I}$ . For any  $d > 0$  and a sufficiently small  $\epsilon$ , the map  $g$  is bounded from below in  $\omega_{I,I}(\epsilon)$  by  $g_{d,I}$ . Since  $g\omega_{I,I}(\epsilon) \subset \omega_{I,I}$  for a small enough  $\epsilon$ , by lemmas 7 and 10 there exists  $\delta_I > 0$  such that

$$|g^k(w, z_1, z_2)| > \delta_I \text{ for any } (w, z_1, z_2) \in \omega_{I,I} \text{ and some } k \geq 0. \quad (83)$$

• We prove now (82) for almost any  $(w, z_1, z_2) \in \omega$ . As in the proof of part (a), the three possibilities (i), (ii) and (iii), listed above, should be considered. We again present a proof for case (iii) only, for which  $a_2 < a_3$ .

Consider the partition

$$\omega_{II} = \cup_{-p < \tilde{p} < p} Q(f_1, f_2, h_{\tilde{p}}),$$

where

$$f_1 \sim z_2^\infty, \quad f_2 \sim z_2^{1/(a_1-a_3)} \quad \text{and} \quad h_{\tilde{p}} = z_2^{1+\tilde{p}}.$$

Each set  $Q(f_1, f_2, h_{\tilde{p}})$  has the boundaries  $q(f_1, h_{\tilde{p}})$  and  $q(f_2, h_{\tilde{p}})$  (see (59) and (60)). By our choice of  $p$  and inequalities (50) and (53),  $gq(f_2, h_{\tilde{p}}; \epsilon_1) \subset \omega_{I,I}$  for a sufficiently small  $\epsilon_1$ . In case (iii), we have for a small enough  $\epsilon_2$

$$gq(f_1, h_{\tilde{p}}; \epsilon_2) \subset \omega_{I,II} \quad \text{and} \quad g^2q(f_1, h_{\tilde{p}}; \epsilon_2) \subset \omega_{II}.$$

Properties (b) and (c) of the map  $g$  imply

$$g^2q(f, h) = q(\tilde{f}, \tilde{h}) \quad \text{with} \quad \tilde{f} \sim z_2^{\tilde{\alpha}}, \quad \tilde{\alpha} = \frac{a_2(1 + \tilde{p} + a_2\alpha) + 1 + a_3\alpha}{a_1(1 + \tilde{p} + a_2\alpha)}, \quad \tilde{h} \sim z_2,$$

for any  $q(f, h) \subset \omega_{II}$  such that  $f \sim z_2^\alpha$ . By part (a) of lemma 6 there exists  $K_{\tilde{p}} > 0$  such that  $g^{K_{\tilde{p}}}q(f_1, h) = q(\hat{f}_1, \hat{h})$  for  $\hat{f}_1 \sim z_2^{\hat{\alpha}}$  and  $\hat{\alpha} < (a_1 - a_3)^{-1}$ . Hence, for a small  $\epsilon_3$   $q^{K_{\tilde{p}}}(f_1, h; \epsilon_3) \subset \omega_{I,I}$ , and by virtue of (62) there exists  $\epsilon_4 > 0$  such that for any  $(w, z_1, z_2) \in \omega_{II}(\epsilon_4)$  at least one of the following inclusions hold:

$$g^{K_{II}}(w, z_1, z_2) \in \omega_{I,I} \quad \text{or} \quad g^k(w, z_1, z_2) \in \omega_{III} \quad \text{for some } k < K_{II}. \quad (84)$$

Here  $K_{II} = \max_{-p \leq \tilde{p} \leq p} K_{\tilde{p}} + 2$ .

Now consider the partition

$$\omega_{III} = \cup_{-p < \tilde{p} < p} Q(f_1, f_2, h_{\tilde{p}}),$$

where

$$f_1 = \left( -\frac{C}{B} z_2 (1 - |z_2|^s) \right)^{1/(a_1-a_3)}, \quad f_2 = \left( -\frac{C}{B} (1 + |z_2|^s) \right)^{1/(a_1-a_3)}, \quad h_{\tilde{p}} = z_2^{1+\tilde{p}}. \quad (85)$$

We have  $gQ(f_1, f_2, h_{\tilde{p}}) = Q(\tilde{f}_1, \tilde{f}_2, \tilde{h})$ , where

$$\tilde{f}_1(z_2) \approx \tilde{f}_2(z_2) \sim z_2^\alpha, \quad \alpha = ((1 + \tilde{p})(a_1 - a_3) + a_2)/a_1 \quad \text{and} \quad -|z_2|^{s(a_1-a_3)/a_1} < \tilde{h} < |z_2|^{s(a_1-a_3)/a_1};$$

$g^2Q(f_1, f_2, h_{\tilde{p}}) = Q(\hat{f}_1, \hat{f}_2, \hat{h})$ , where

$$\hat{f}_1(z_2) \sim z_2^\infty, \quad \hat{f}_2(z_2) \sim z_2^{\hat{\alpha}}, \quad \hat{\alpha} = s(a_1 - a_3)a_1^{-2}\alpha^{-1} + a_2a_1^{-1} \quad \text{and} \quad h \sim z_2.$$

(This can be checked by substituting the expressions for  $f_1, f_2$  and  $h_{\tilde{p}}$  (85) into (41).)

For  $1 \leq k \leq K = K_{II}/2 + 1$ , denote

$$H_{\bar{p},k} = g^{-2k}(g^{2k}Q(f_1, f_2, h_{\bar{p}}) \cap \omega_{III}) \cap \omega_{III}, \quad (86)$$

$$H_k = \cup_{-p < \bar{p} < p} H_{\bar{p},k}. \quad (87)$$

Note that  $H_{\bar{p},k} = Q(f_{k_1}, f_{k_2}, h_{k,\bar{p}})$ , where

$$f_{k_1} + CB^{-1}z_2 \sim f_{k_2} + CB^{-1}z_2 \sim z_2^{\alpha_k}, \quad \alpha_k > s, \quad \text{and } f_{k_1} - f_{k_2} \sim z_2^{\gamma_k}, \quad \gamma_k > \alpha_k.$$

By virtue of (84), there exists  $\epsilon_5 > 0$  such that for any  $(w, z_1, z_2) \in \omega_{III}(\epsilon_5)$  at least one of the following inclusions hold true:

$$g^{2K}(w, z_1, z_2) \in \omega_{I,I} \text{ or } g^k(w, z_1, z_2) \in \omega_{III} \text{ for some } k < 2K. \quad (88)$$

For a  $\delta > 0$  define the sets

$$H_k(\delta) = H_k \cap B_\delta, \quad (89)$$

$$H_{k_1, k_2, \dots, k_N}(\delta) = g^{-k_1} H_{k_2, \dots, k_N}(\delta) \cap H_{k_2, \dots, k_N}(\delta) \quad (90)$$

and

$$G_N(\delta) = \cup_{1 \leq k_1, \dots, k_N \leq K} H_{k_1, \dots, k_N}(\delta).$$

Let  $\epsilon_6 > 0$  be the  $\epsilon$  whose existence is asserted in lemma 5 for  $d = 1/2$ . By this lemma, for any  $\delta < \epsilon_6$

$$\mu_{z_1}(G_1(\delta)) < \frac{1}{2} \mu_{z_1}(\omega_{III}(\delta)).$$

Applying further lemma 5 (note that  $C$ ,  $B_k$  and  $c_k$  are independent of  $N$  and  $\beta_m = \beta_1$ ), we obtain

$$\mu_{z_1}(\cup_{1 \leq k_1 \leq K} H_{k_1, k_2, \dots, k_N}(\delta)) < \frac{1}{2} \mu_{z_1}(H_{k_2, \dots, k_N}(\delta)),$$

which implies

$$\mu_{z_1}(G_N(\delta)) < \frac{1}{2} \mu_{z_1}(G_{N-1}(\delta)).$$

Hence,

$$\mu_{z_1}(G_N(\delta)) < \frac{1}{2^N} \mu_{z_1}(\omega_{III}(\delta)) \Rightarrow \lim_{N \rightarrow \infty} \mu(G_N(\delta)) = 0.$$

Denote

$$G_\infty(\delta) = \lim_{N \rightarrow \infty} G_N(\delta)$$

and set  $\delta_{II} = \min_{1 \leq j \leq 6}(\epsilon_j)$ . By definition of sets  $G_N(\delta)$ ,

$$\begin{aligned} & \text{if } (w, z_1, z_2) \in \omega_{III} \setminus G_\infty(\delta_{II}) \text{ and } g^k(w, z_1, z_2) \in B_{\delta_{II}} \text{ for all } k \geq 0, \\ & \text{then there exists } l > 0 \text{ such that } g^l(w, z_1, z_2) \in \omega_{I,I}. \end{aligned} \quad (91)$$

Therefore, by (62) and (83)

$$\text{for any } (w, z_1, z_2) \in \omega \setminus G_\infty(\delta_{III}), \quad |g^k(w, z_1, z_2)| > \delta_{III} \text{ for some } k \geq 0, \quad (92)$$



where  $\delta_{III} = \min(\delta_I, \delta_{II})$ .

- Finally, we consider an arbitrary  $(w, z_1, z_2)$ . Let  $f_{III}(z_2)$  and  $h_{III}(z_2)$  be functions solving the system

$$Bf^{a_1} + Cf^{a_3}z_2 = h(Df^{a_1} + Ef^{a_3}z_2), \quad Af^{a_2}h = f(Df^{a_1} + Ef^{a_3}z_2),$$

which is equivalent to  $g$ -invariance of  $q(f_{III}, h_{III})$ . There exists a small  $\epsilon_7 > 0$  such that

$$(i) \text{ either } q(f_{III}, h_{III}; \epsilon_7) \subset \Omega_{III} \text{ or } (ii) \text{ } q(f_{III}, h_{III}; \epsilon_7) \cap \Omega_{III} = \emptyset,$$

depending on the values of  $a_1, a_2$  and  $a_3$ .

Lemma 6(c) and property (d) of the map  $g$  imply that in case (i) there exists  $\epsilon_8 > 0$  such that for any  $(w, z_1, z_2) \in \Omega_{III}(\epsilon_8) \setminus q(f_{III}, h_{III})$  we can find  $k > 0$  such that  $g^k(\mathbf{x}) \notin \Omega_{III}$ .

Define

$$\mathcal{G}_\infty(\delta) = \cup_{0 \leq k < \infty} g^{-k}G_\infty(\delta) \cup q(f_{III}, h_{III}; \delta_{III})$$

and note that  $\mu(\mathcal{G}_\infty(\delta_{III})) = 0$ . Set  $\delta_0 = \min(\delta_{III}, \epsilon_7, \epsilon_8)$ . By (61), (62), (92) and definition of the set  $\mathcal{G}_\infty(\delta)$ ,

$$\text{for any } (w, z_1, z_2) \notin \mathcal{G}_\infty(\delta_0), \quad |g^k(w, z_1, z_2)| > \delta_0 \text{ for some } k \geq 0. \quad (93)$$

Since  $\mu(\mathcal{G}_\infty(\delta_0)) = 0$ , for  $a_2 > 0$  and  $a_3 > 0$  statement (b) is proven.

(b.2) Next, we assume that  $a_2 > 0$  and  $a_3 < 0$ . By simple algebra (not presented), the latter inequality together with (50) imply  $\lambda^I < 1$ .

- By the same arguments as in part (b.1), there exists  $\delta_I > 0$  such that for any  $(w, z_1, z_2) \in \omega_{I,I}$  the statement (83) holds true.

- Consider the partition

$$\omega_{II} = \left( \cup_{-p < \tilde{p} < p} Q(f_1, f_2, h_{\tilde{p}}) \right) \cup \left( \cup_{-p < \tilde{p} < p} Q(f_3, f_4, h_{\tilde{p}}) \right)$$

where

$$f_1 = -|z_2|^\infty, \quad f_2 = -|z_2|^{-1/(a_1-a_2)}, \quad f_3 = -|z_2|^{-1/(a_1-a_2)}, \quad f_4 = -|z_2|^{-1/(a_1-a_3)}, \quad h = z_2^{1+\tilde{p}}.$$

There exists  $\delta_{II} > 0$  such that

$$|g(w, z_1, z_2)| > \delta_{II}$$

for any  $(w, z_1, z_2) \in Q(f_1, f_2, h_{\tilde{p}}, \delta_{II})$ . By part (c) of lemma 6, for any  $Q(f_3, f_4, h_{\tilde{p}})$  there exists  $f_5(z_2)$  such that  $gq(f_5, h_{\tilde{p}}) = q(f_5, h_{\tilde{p}})$ , and any  $(w, z_1, z_2) \in \omega_{II}(\epsilon_1) \setminus q(f_5, h_{\tilde{p}})$  satisfies

$$|g^k(w, z_1, z_2)| \notin \omega_{II}$$

for some  $k > 0$ .

Define the sets  $H_{\mathbf{k}}$  by (86)-(90),

$$G_N(\delta) = \cup_{1 \leq k_1, \dots, k_N < \infty} H_{k_1, \dots, k_N}(\delta) \quad (94)$$

and

$$G_\infty(\delta) = \lim_{N \rightarrow \infty} G_N(\delta). \quad (95)$$

By the arguments employed in part (b.1),  $\mu(G_\infty) = 0$  and any  $(w, z_1, z_2) \in \omega_{III} \setminus G_\infty$  has the property (91); consequently, (92) holds true for some  $\delta_{III} > 0$ .

• Define

$$\mathcal{G}_\infty(\delta) = \cup_{0 \leq k < \infty} g^{-k} G_\infty(\delta). \quad (96)$$

(now  $q(f_{III}, h_{III})$  is not included into this set due to the bound

$$|g(w, z_1, z_2)| \sim |(w, z_1, z_2)|^{a_1/(a_1-a_3)}$$

for  $(w, z_1, z_2) \in q(f_{III}, h_{III})$  and  $a_1/(a_1 - a_3) < 1$ ). For any  $(w, z_1, z_2) \in B_\epsilon \setminus \mathcal{G}_\infty(\delta_0)$  (where  $\delta_0 = \min(\delta_{III}, \epsilon_8)$  and  $\epsilon_8$  is defined in part (b.1) of the proof), the inequality  $|g^l(w, z_1, z_2)| > \delta_0$  is satisfied for some  $l \geq 0$ .

(b.3) Finally, we assume  $a_2 < 0$ . Again, we study the sets  $g\omega_J$  and  $g\Omega_J$ . The key points of the proof follow (details are omitted):

• By (63) and lemma 6(b), there exist  $\delta_{II} > 0$  and  $K_{II} > 0$  such that

$$|g^k(\mathbf{x})| > \delta_{II} \text{ for some } 0 < k < K_{II}, k \geq 0$$

for any  $\mathbf{x} \in \omega_{II}$ .

• By property (b) of the map  $g$  and lemma 6(c) there exist a set

$$\tilde{q} = \cup_{-p \leq \tilde{p} \leq p} q(f_{\tilde{p},0}, h_{\tilde{p}})$$

and  $\epsilon_I > 0$  such that for any  $(w, z_1, z_2) \in \omega_I(\epsilon_I) \setminus \tilde{q}$  we can find  $k > 0$  such that

$$g^k(w, z_1, z_2) \notin \omega_I(\epsilon_I).$$

Consider the partition

$$\omega_{IV} = \cup_{-p < \tilde{p} < p} Q(f_1, f_2, h_{\tilde{p}}),$$

where

$$f_1 = \left( -\frac{E}{F} z_2 (1 - |z_2|^s) \right)^{1/(a_1-a_3)}, \quad f_2 = \left( -\frac{E}{F} (1 + |z_2|^s) \right)^{1/(a_1-a_3)}, \quad h_{\tilde{p}} = z_2^{1+\tilde{p}}.$$

For  $2 \leq k \leq \infty$  define the sets

$$H_{\tilde{p},k} = g^{-2k}(g^{2k}Q(f_1, f_2, h_{\tilde{p}}) \cap \omega_{IV}) \cap \omega_{IV} \quad (97)$$

and the sets  $H_k$ ,  $H_k(\delta)$ ,  $H_{\mathbf{k}}$ ,  $G_N(\delta)$  and  $G_\infty(\delta)$  by (87), (89), (90), (94) and (95), respectively. There exists  $\delta_I > 0$  such that for any  $(w, z_1, z_2) \in \omega \setminus (G_\infty(\delta_I) \cup \tilde{q})$

$$|g^l(w, z_1, z_2)| > \delta_I \text{ for some } l \geq 0.$$

• Define the set  $\mathcal{G}_\infty(\delta)$  by (96) and note that for any  $(w, z_1, z_2) \notin (\mathcal{G}_\infty(\delta_I) \cup \tilde{q})$

$$|g^l(w, z_1, z_2)| > \delta_{III} \text{ for some } l \geq 0,$$

where  $\delta_{III} < \min(\delta_I, \delta_{II}, \epsilon_I)$ . Since  $\mu(\mathcal{G}_\infty(\delta_I)) = 0$ , the theorem is proven. **QED**

**Theorem 5** *Consider the map*

$$g(w, z_1, z_2) = (Az_1|w|^{a_2}, Bw^{a_1} + Cz_2|w|^{a_3}, Dw^{a_1} + Ez_2|w|^{a_3}), \quad (98)$$

*where  $a_1 > 0$  and  $a_2(a_1 - a_3) + (a_1 - a_3)^2 - a_1 > 0$ . Then*

*(a) The fixed point  $(w, z_1, z_2) = 0$  of the map  $g$  is asymptotically stable if and only if*

$$a_2 > 0 \text{ and } a_3 > 0.$$

*(b) The fixed point  $(w, z_1, z_2) = 0$  of the map  $g$  is fragmentarily asymptotically stable if and only if*

$$a_2 > -1 \text{ and } a_3 > 0.$$

The proof is based on lemmas 7 and 8, as the proof of theorem 4 is based on lemma 7.